Norwegian University of Science and Technology
Department of Mathematics

TMA4145 Linear Methods
Autumn 2012

Exercise set 7

This week's exercises are all from old exams (it's good practice!). The dates refer to which exams they were taken from.

1 *(Exercise 1, December 2005)* Let (X,d) be a metric space, and let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences in (X,d), both converging towards $a \in X$. Let $\{z_n\}_{n\in\mathbb{N}}$ be the sequence defined by

$$z_n = \begin{cases} x_{\frac{n+1}{2}} & n \text{ is odd}, \\ y_{\frac{n}{2}} & n \text{ is even}, \end{cases}$$

i.e., $\{z_n\}_{n\in\mathbb{N}} = (x_1, y_1, x_2, y_2, \ldots)$. Show that $\{z_n\}_{n\in\mathbb{N}}$ is Cauchy in (X,d).

2 Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be a real $n \times n$ matrix, and consider the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $x \mapsto Ax$.

a) *(Exercise 2a, November 2001)*
Show that

$$\|Ax\|_{l_\infty} \leq (\max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|)\|x\|_{l_\infty}.$$

What is $\|T\|$ (given that $\mathbb{R}^n$ is equipped with the maximum norm)?

b) *(Exercise 2a, December 2003)*
Show that

$$\|Ax\|_{l_1} \leq (\max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|)\|x\|_{l_1}.$$

3 *(Exercise 3, November 2001)* Find the LU-decomposition of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

for which the diagonal elements of L are all ones.

4 *(Exercise 2a, December 2005)* The matrix A has an LU-decomposition as follows:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find bases for the subspaces $\operatorname{ran}(A)$ and $\ker(A^T)$ of $\mathbb{R}^3$; $\ker(A)$ and $\operatorname{ran}(A^T)$ of $\mathbb{R}^5$.

5 *(Exercise 3, December 2000)* Let A be a real $m \times r$-matrix such that its column vectors are linearly independent, and B be a real $r \times n$-matrix such that its row vectors are linearly independent. Show that:

a) $\ker(AB) = \ker(B)$.

b) $\operatorname{ran}(AB) = \operatorname{ran}(A)$.

6 *(Challenge. Exercise 5, November 2002)*

Let $v_1, v_2 \colon \mathbb{R} \to \mathbb{R}$ be defined by $v_1(t) = e^t$, $v_2(t) = e^{-t}$, and consider the vector space $V = \{av_1 + bv_2 : a, b \in \mathbb{R}\}$.

Let A be the matrix realization of a linear transformation $T \colon V \to V$ with respect to the basis $\{v_1, v_2\}$, and let B be its matrix realization with respect to the basis $\{\cosh, \sinh\}$. Find an invertible matrix S such that

$$A = S^{-1}BS.$$

Finally, calculate A and B when T is the derivation operator $D = \frac{d}{dt}$. Check that in this case $A = S^{-1}BS$.

Tip:

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \qquad \sinh(t) = \frac{e^t - e^{-t}}{2}$$