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Department of Mathematics

TMA4145 Linear Methods
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Exercise set 8

1 Which of the following mappings are (i) linear, (ii) linear and bounded? Justify your answers.

a)
$$T\colon \mathbb{R} \to \mathbb{R}, \qquad t^3 \mapsto 3t^2,$$
with $\|\cdot\|_{\mathbb{R}} = |\cdot|$.

b)
$$T\colon P_n(\mathbb{R}) \to P_n(\mathbb{R}), \qquad \sum_{j=0}^{n} a_j x^j \mapsto \sum_{j=1}^{n} j a_j x^{j-1},$$
with $\|\sum a_j x^j\|_{P_n(\mathbb{R})} = \|(a_0, \ldots, a_n)\|_{\mathbb{R}^{n+1}}$.

c)
$$C^1([0,1],\mathbb{R}) \cap BC([0,1],\mathbb{R}) \to BC([0,1],\mathbb{R}), \qquad f \mapsto f',$$
with the $BC([0,1],\mathbb{R})$-norm on the dense subspace $C^1([0,1],\mathbb{R})$ of $BC([0,1],\mathbb{R})$.[1]

d)
$$BC(\mathbb{R}) \to \mathbb{R}, \qquad f \mapsto f(1)$$
with the usual Euclidean norm on $\mathbb{R}$ (as in (a)).

2 *(Variant of Problem 4, 2007, Problem 3, 2008, and possibly others)* Let
$$l_0 = \{\{x_j\}_{j\in\mathbb{N}} \colon \exists N \in \mathbb{N},\ x_j = 0 \text{ for all } j \geq N\}$$
be the space of sequences with finitely many non-zero entries. In all of this problem you may assume that the sequences are real, or complex, as you like.

a) Show that $\overline{l_0} = l_2$ in l_2.

b) Let
$$c_0 := \{\{x_j\}_j \in l_\infty \colon \lim_{j\to\infty} x_j = 0\}$$
be the subspace of l_∞ consisting of sequences with a vanishing limit. Show that $\overline{l_0} = c_0$ in l_∞.

c) Finally, show that $\overline{c_0} \subsetneq l_\infty$ in l_∞.

Hint: You may find (b) and (c) easier than (a).

[1] Recall that a real-valued continuous function attains its maximum/minimum on a compact set, so that $C([0,1],\mathbb{R}) = BC([0,1],\mathbb{R})$ as sets and linear spaces. The notation $BC([0,1],\mathbb{R})$ is to emphasize that this space is equipped with the supremum norm. Note also, as stated above, that $C^1([0,1],\mathbb{R})$ is not a closed subspace of $BC([0,1],\mathbb{R})$; its closure (and completion with respect to the BC-norm) is all of $BC([0,1],\mathbb{R})$. This is a consequence of the Stone–Weierstrass theorem.

3 *(Variant of Kreyszig 2.10:10)* Find an unbounded linear transformation $l_\infty(\mathbb{R}) \to \mathbb{R}$.

4 *(Kreyszig 2.10:15)* The *annihilator*, M^α, of a non-empty subset $M \subset X$ of a normed space is the linear subspace of the dual X' consisting of those bounded linear functionals that vanish on M:

$$M^\alpha \stackrel{def.}{=} \{T \in X' \colon Tx = 0 \text{ for all } x \in M\}.$$

What is the annihilator M^α of

$$M = \{(1,0,-1),(1,-1,0),(0,1,-1)\} \subset \mathbb{R}^3?$$

Hint: recall that $B(\mathbb{R}^3,\mathbb{R}) \cong \mathbb{R}^3$. Each bounded linear functional on $\mathbb{R}^3$ is given by a dot product $(x_1,x_2,x_3) \mapsto \sum_{j=1}^3 x_j y_j$. Thus M^α can be identified with a subspace of $\mathbb{R}^3$.

5 *(Challenge, Kreyszig 2.10:8)* For $q, p \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, l_p is the dual of l_q. For $p = 1, q = \infty$, this is not true: l_∞ is the dual of l_1, but l_1 is not the dual of l_∞. Show that l_1 is the dual of $c_0 \subset l_\infty$.