TMA4145 Linear Methods
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Exercise set 9

1 (*Compositions of operators*)

a) Show that the composition $S \circ T$ of two bounded operators $T \in B(X,Y)$ and $S \in B(Y,Z)$ is a bounded operator $X \to Z$.[1]

b) What is $A \circ B$ when $A \in M_{m\times r}$ and $B \in M_{r\times n}$ are given by matrices (real or complex)?

c) Find an example with $X = Y = Z$ showing that $T \circ S$ need not equal $S \circ T$.

d) Find an example with X, Y, Z all different for which

$$\|S \circ T\|_{B(X,Z)} < \|S\|_{B(Y,Z)} \|T\|_{B(X,Y)}.$$

2 (*Kreyszig 5.4:2*) Let $[a,b] \subset \mathbb{R}$ be closed and bounded, $v \in C([a,b],\mathbb{R})$ a continuous function on $[a,b]$, and $k \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$ a Lipschitz-continuous function such that

$$|k(t,\tau,x_1) - k(t,\tau,x_2)| \leq L|x_1 - x_2| \quad \text{for all} \quad t,\tau \in [a,b], \quad x_1, x_2 \in \mathbb{R},$$

where L is a fixed constant. Show that the nonlinear integral equation

$$x(t) - \mu \int_a^b k(t,\tau,x(\tau))\, d\tau = v(t), \qquad t \in [a,b],$$

has a unique solution $x \in C([a,b],\mathbb{R})$ for any real constant μ with $|\mu| < \frac{1}{L(b-a)}$.

3 (*Kreyszig 5.3:8*)

a) Find the first three terms x_1, x_2, x_3 in the Picard iteration for

$$x' = 1 + x^2, \qquad x(0) = 0.$$

b) Verify that the terms involving $t, \ldots, t^5$ are in agreement with the Maclaurin expansion for the exact solution (you should find the solution as part of the problem; its series expansion you can look up).[2]

[1] This gives $B(X) = B(X,X)$ the additional structure of an *algebra*: not only can we add and scale elements in $B(X)$, but also compose them (since $S \circ T \in B(X)$ whenever $S, T \in B(X)$). The study of such operator algebras is an important field of mathematics.

[2] A Maclaurin expansion is a Taylor expansion at the origin.

4 (*Problem 2, 2000*)

a) Let $T \in L(BC([0,1],\mathbb{R}))$ be the linear transformation determined by

$$(Tx)(t) = \int_0^1 ts\, x(s)\, ds.$$

Show that: i) T is bounded, ii) $\|T\| = \frac{1}{2}$.

b) Find $x \in C([0,1],\mathbb{R})$ that satisfies

$$x(t) = 4 + \int_0^1 ts\, x(s)\, ds, \qquad t \in [0,1].$$

5 (*Problem 2, 2003*) Let $A = (a_{ij})_{ij} \in M_{n\times n}(\mathbb{C})$ and $b \in \mathbb{C}^n$. Prove that the system

$$x = Ax + b, \qquad x \in \mathbb{C}^n,$$

can be solved by iteration if $\sum_{j=1}^n |a_{jk}| < 1$ for all $k = 1, \dots, n$.

Hint: Problem 2, Set 7.

6 (*Challenge*)

a) Pick a suitable basis for $P_2(\mathbb{R})$ and express the linear operator

$$T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R}), \qquad Tp(x) = p(x) + xp'(x),$$

as a matrix (in that basis). Determine $\ker(T)$ to conclude that T is an isomorphism from $P_2(\mathbb{R})$ onto itself. For which p is $Tp = p$?

b) Consider then the transformation

$$T\colon P(\mathbb{R}) \to P(\mathbb{R}), \qquad Tp(x) = \sum_{j\geq 0} x^j p^{(j)}(x).$$

Is it linear? Injective? An isomorphism on $P(\mathbb{R})$? For which $p \in P(\mathbb{R})$ is $Tp = p$?

c) Identify $P(\mathbb{R})$ with $l_0(\mathbb{R}) \subset l_\infty(\mathbb{R})$ (cf. Problem 2, set 8) via

$$\sum_{j=0}^n a_j x^j \sim (a_0, \dots, a_n, 0, \dots) \quad \text{for} \quad p \in P_n(\mathbb{R}),$$

and endow $P(\mathbb{R}) \cong l_0(\mathbb{R})$ with the l_∞-norm. Show that, for this choice of norm, $T\colon P(\mathbb{R}) \to P(\mathbb{R})$ is well-defined as a map between normed spaces, but unbounded.