Motivation: The pendulum

Pendulum equation:

$$\ddot{x} + \omega^2 \sin x = 0. \tag{1}$$

No solution in terms of elementary functions.

Phase plane analysis:

1. Equivalent system ($y = \dot{x}$):

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\omega^2 \sin x. \end{cases} \tag{2}$$

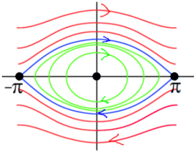
2. Implicit solution = phase trajectory (here C is energy):

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -\frac{\omega^2 \sin x}{y} \quad \Longrightarrow \quad \frac{1}{2}y^2 - \omega^2 \cos x = C = \text{const.}$$ i.e. a curve for each C.

3. Equilibrium points (x_e, y_e) = constant solutions of (2):

$$\Leftrightarrow \; (y_e, -\omega^2 \cos x_e) = (0,0) \Leftrightarrow \; (x_e, y_e) = (n\pi, 0), \; n \in \mathbb{Z}.$$

4. Phase diagram: All phase trajectories (periodic in x, symmetric about $y = 0$)



5. Interpretation: Red = rotations, green = swinging back and forth.

Motivation: The pendulum

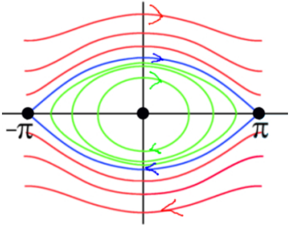
Stabililty:

A solution is stable if all solutions starting "near" remain "near" the solution for all times.

Stability for the pendulum:

Since solutions follow phase trajectories in the direction of the arrows, we see from the phase portrait that:

1. The equilibrium points $(2m\pi, 0)$ are stable.
2. The equilibrium points $((2m+1)\pi, 0)$ are unstable.
3. The separatrices (blue curve) are unstable.
4. For all other solutions more powerful methods are needed!



Due to small disturbances, physical systems tend over time to be (near) their stable equilibrium solutions (not neccesarily equilibrium points).

Linear 2×2 systems of ODEs

Initial value problem:

$$(1) \qquad \begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 \\ \dfrac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2, \end{cases} \quad \text{or} \quad \frac{d\vec{x}}{dt} = A\vec{x} \quad \text{or} \quad \frac{dx}{dt} = Ax,$$

$$(2) \qquad \vec{x}(0) = \vec{x}_0.$$

where $A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$ = constant and $\vec{x} = x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$.

Autonomous equation (does not depend on t) in normal form (like $\dot{x} = f(x,t)$).

Results:

1. Theorem 1: There is only one solution of (1) and (2) for all $t \in \mathbb{R}$.
2. Theorem 2: x_1 and x_2 solve (1) $\quad \Rightarrow \quad$ $c_1\vec{x}_1 + c_2\vec{x}_2$ solve (1) for all $c_1, c_2 \in \mathbb{R}$.
3. Theorem 3: x_1 and x_2 solve (1) and are linear independent
   $\Rightarrow$ any solution of (1) can be written as $c_1\vec{x}_1 + c_2\vec{x}_2$ for some $c_1, c_2 \in \mathbb{R}$.

In Tm. 3 x_1, x_2 is a basis, and $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$ a general solution of (1).

Solving equation (1)

$$(1) \qquad \frac{d\vec{x}}{dt} = A\vec{x} \qquad \text{where} \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \text{constant}.$$

Idea: Find a basis of two independent solutions.

Test solution: $\vec{x} = \vec{v}e^{\lambda t}$, $\vec{v} \neq \vec{0}$.

1. Solves (2) iff $(A - \lambda I)\vec{v} = \vec{0}$ ... Eigenvalue problem.
2. Non-zero solutions $\vec{v}$ iff characteristic equation holds

$$\det(A - \lambda I) = \lambda^2 - p\lambda + q = 0, \quad \text{where} \quad \begin{cases} p = \operatorname{tr} A = a_{11} + a_{22}, \\ q = \det A = a_{11}a_{22} - a_{12}a_{21}. \end{cases}$$

3. Hence always solutions $(\lambda_1, \vec{v}_1)$, $(\lambda_2, \vec{v}_2)$, but complex solutions possible.

Solving linear 2x2 systems

(1) $\qquad \dot{x} = Ax \qquad$ or $\qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Test solution: $x = ve^{\lambda t}$, $v \neq 0$.

1. Solves (1) iff $(A - \lambda I)v = 0$ ... Eigenvalue problem ... solutions iff:

   $\det(A-\lambda I) = \lambda^2 - p\lambda + q = 0, \quad q = a_{11} + a_{22}, \quad q = a_{11}a_{22} - a_{12}a_{21}.$

2. Hence always solutions (λ_1, v_1), (λ_2, v_2). All possible cases:
   1. $\lambda_1 \neq \lambda_2$ real: $\quad v_1, v_2 \in \mathbb{R}^2$ are linearly independent.
   2. $\lambda_1 = \lambda_2$ real: $\quad v_1, v_2 \in \mathbb{R}^2$ may or may not be lin. independent.
   3. $\lambda_1 = \bar{\lambda}_2$ complex: $v_1 = \bar{v}_2 \in \mathbb{C}^2$ are linearly independent.

General solution: $x = C_1 x_1 + C_2 x_2, \quad C_1, C_2 \in \mathbb{R},$ where

1. λ_1, λ_2 real, v_1, v_2 lin. independent: $\boxed{x_1(t) = v_1 e^{\lambda_1 t}}\boxed{x_2(t) = v_2 e^{\lambda_2 t}}$
2. $\lambda_1 = \lambda_2$ real, only one lin. indep. v: $\boxed{x_1(t) = ve^{\lambda_1 t}}$ and

   $\boxed{x_2 = (vt + u)e^{\lambda_1 t}}$ where $(A - \lambda_1 I)u = v$.
3. $\lambda_1 = \bar{\lambda}_2 = \alpha - i\beta$ complex: $\boxed{x_1(t) = \mathrm{Re}(v_1 e^{\lambda_1})}\boxed{x_2(t) = \mathrm{Im}(v_1 e^{\lambda_1})}$

Linear 2x2 systems

$$(1) \qquad \dot{x} = Ax \quad \text{or} \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

1. Thm: All solutions of (1) can be written in the form

   $$x(t) = C_1 x_1(t) + C_2 x_2(t),$$

   where x_1 and x_2 are defined as above in each case.

2. Thm: All solutions of (1) are linear combinaitions of products of trigonometric, exponential, and plynomial functions.

3. Thm: The initial value problem (1) and

   $$(2) \qquad x(0) = x_0,$$

   has a solution for all $x_0 \in \mathbb{R}^2$ and all $t \in \mathbb{R}$.

Fundamental matrix and Stability

$$\dot{x}(t) = Ax(t), \tag{1}$$
$$x(0) = x_0. \tag{2}$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is constant and $x \in \mathbb{R}^2$.

Concepts:

1. Fundamental matrix: $\Phi(t) = [x_1(t), x_2(t)]$ where x_1, x_2 basis for (1)
2. Flow ϕ: $x(t) = \phi(t; x_0)$ solve (1) and (2).
3. Lemma: $\phi(t; x_0) = \Phi(t)\Phi(0)^{-1}x_0$.

Stability: Let $x(t) = \phi(t; x_0)$.

1. $x(t), t > 0$, stable if for all $\varepsilon > 0$ there is $\delta > 0$ such that
$$|x_1 - x_0| < \delta \quad \Rightarrow \quad |\phi(t; x_1, t_0) - \phi(t; x_0, t_0)| < \varepsilon \quad \text{for all } t > 0.$$
2. $x(t), t > 0$, asymptotic stable if stable and there is $\eta > 0$ s.t.
$$|x_1 - x_0| < \eta \quad \Rightarrow \quad |\phi(t; x_1, t_0) - \phi(t; x_0, t_0)| \to 0 \text{ as } t \to \infty.$$

Obs: The definitions hold for general non-linear $n \times n$ systems!

Stability of solutions of linear systems

$$\dot{x}(t) = Ax(t), \tag{1}$$
$$x(0) = x_0. \tag{2}$$

where $A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$ is constant and $x \in \mathbb{R}^2$.

Theorem 1: Let Φ be any fundamental matrix of (1).

1. $\|\Phi(t)\| \leq M < \infty$ for all $t \geq 0 \quad \Rightarrow \quad$ all solutions of (1) are stable.
2. $\lim_{t\to 0} \|\Phi(t)\| = 0 \quad \Rightarrow \quad$ all solutions of (1) are asymptot. stable.
3. $\lim_{t\to 0} \|\Phi(t)\| = \infty \quad \Rightarrow \quad$ all solutions of (1) are unstable.

Theorem 2: Let λ_1, λ_2 be the eigenvalues of A.

1. all solutions of (1) are stable $\quad \Rightarrow \quad \max_i \mathrm{Re}(\lambda_i) \leq 0$.
2. $\max_i \mathrm{Re}(\lambda_i) \leq 0$ and $\lambda_1 \neq \lambda_2 \quad \Rightarrow \quad$ all solutions of (1) are stable.
3. $\max_i \mathrm{Re}(\lambda_i) < 0 \quad \Rightarrow \quad$ all solutions of (1) are asymptot. stable.
4. $\max_i \mathrm{Re}(\lambda_i) \geq 0 \quad \Rightarrow \quad$ all solutions of (1) are unstable.

Autonomous linear 2x2 systems

(1) $$\dot{x} = Ax; \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Equilibrium points:

- Constant solutions x_e of (1), i.e. solutions of $Ax_e = 0$.
- $\lambda_1, \lambda_2 \in \mathbb{R}$: Node if $\lambda_1\lambda_2 > 0$, and saddle if $\lambda_1\lambda_2 < 0$.
- $\lambda_1 = \bar{\lambda}_2 \in \mathbb{C}$: Spiral if $\mathrm{Re}\,\lambda_1 \neq 0$, and center if $\mathrm{Re}\,\lambda_1 = 0$.
- $\lambda_1 = \lambda_2$ or either $\lambda_1 = 0$ or $\lambda_2 = 0$: Degenerate cases.

Phase diagrams/portraits:

1. Phase trajectory through x_0: $\{x(t) : t \in \mathbb{R}, x \text{ solves (1)}, x(0) = x_0\}$

   Tangent at x: Ax $[= \dot{x}]$.

   Direction: Direction of x as time increases = tangent direction

   - Direction in one pt. $\to$ all directions by continuity of directions

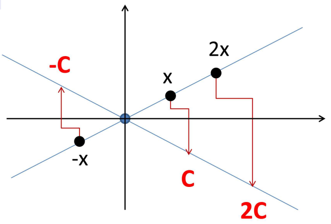
   Equation: $\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{a_{21}x_1 + a_{22}x_2}{a_{11}x_1 + a_{12}x_2}$

   Equilibrium point: Trajectory = one point

2. Phase plane/diagram: x_1x_2-plane/sketch of "all" phase trajectories.

Remark: One trajectory through every $x_0 \in \mathbb{R}^2$; trajectories do not cross!

Scaling and symmetry for linear systems



Scaling: $s \in \mathbb{R}, \neq 0$

1. Point x: sx
2. Curve C: $sC = \{sx : x \in C\}$

Observe:

1. $\dot{x} = Ax \underset{y=sx}{\Leftrightarrow} \dot{y} = Ay$ (scale invariant)
2. C part of trajectory through x_0 $\Rightarrow$ sC part of trajectory through sx_0

Consequences:

1. In any sector from 0:
   1. One trajectory determines the whole diagram by scaling.
   2. Equal transit time for all trajectory-segments in sector (check it!)
2. Phase diagram symmetric about 0.
3. No isolated closed cuves.

Read for yourself Jordan-Smith chp. 2.6

Linear non-autonoumous nxn systems

$$(1)\quad \begin{cases} \dot{x} = A(t)x + b(t), \\ x(t_0) = x_0, \end{cases} \qquad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

We always assume: A, b continuous for all $t \in \mathbb{R}$.

Results and definitions:

1. Non-homogeneous, non-autonomous equation in normal form.
2. There exists a unique solution of (1) for all $x_0 \in \mathbb{R}^n$ and $t \in \mathbb{R}$.
3. Flow $\phi(t; x_0, t_0)$: The solution of eq'n $(1)_1$ and initial data $(1)_2$.

Homogeneous equation $b \equiv 0$:

1. Basis: n linearly independent solutions $x_1, \ldots, x_n$ of $\dot{x} = Ax$
2. Fundamental matrix $\Phi \in \mathbb{R}^{n \times n}$, a for all $t \in \mathbb{R}$ invertible solution of
$$\dot{\Phi} = A\Phi.$$
3. **OBS:** $\Phi = [x_1, \ldots, x_n]$ and $b \equiv 0 \Rightarrow \phi(t; x_0, t_0) = \Phi(t)\Phi(t_0)^{-1}x_0$
4. There always exists a fundamental matrix (and a basis) for (1).

Linear $n \times n$ systems

Non-autonomous equation $\dot{x} = A(t)x + b(t)$; $A \in \mathbb{R}^{n\times n}$.

$x(t) = \Phi(t)\Phi^{-1}(t_0)x(t_0) + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)b(s)ds$ if Φ fund. matrix.

Autonomous equation $\dot{x} = Ax$; A constant, eigenval's/-vec's λ_i/r_i.

1. $x = re^{\lambda t}$ solve (1) iff $(A - \lambda I)r = 0$.

   Basis when n lin. independent r_i: $x_1 = r_1e^{\lambda_1 t}, \dots, x_n = r_ne^{\lambda_n t}$.

2. Basis in general: There are polynomial p_i (order $\leq n$) such that

$$x_1(t) = p_1(t)e^{\lambda_1 t}, \dots, x_n(t) = p_n(t)e^{\lambda_n t}.$$

3. Real basis: $\mathrm{Re}\, x_i$, $\mathrm{Im}\, x_i$, $i = 1, \dots, n$ (give n lin. indep. solutions).

Jordan form and exponential

Jordan form: $A \in \mathbb{C}^{n\times n}$

1. Eigenvalues/-vectors λ_i / r_i.
2. Complex diagonalization: When n lin. independent r_i's,
$$A = P\Lambda P^{-1} \quad \text{where} \quad \Lambda = \operatorname{diag}(\lambda_i) \in \mathbb{C}^{n\times n}, \quad P = [r_1 \dots r_n].$$
3. Jordan form: For any real matrix A! There is a P s.t.
$$A = P^{-1}JP \quad \text{where} \quad J = \operatorname{diag}(B_1, \dots, B_m) \in \mathbb{R}^{n\times n},$$

$$B_i = \ \lambda_i, \ \begin{pmatrix} \mathrm{Re}\,\lambda_i & \mathrm{Im}\,\lambda_i \\ -\mathrm{Im}\,\lambda_i & \mathrm{Re}\,\lambda_i \end{pmatrix} =: D_i, \ \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ 0 & & & \lambda_i \end{bmatrix}, \text{ or } \begin{bmatrix} D_i & I & & 0 \\ & \ddots & \ddots & \\ & & D_i & I \\ 0 & & & D_i \end{bmatrix}.$$

Matrix exponential: $A \in \mathbb{C}^{n\times n}$

1. $e^A := \lim_{n\to\infty} \left(I + A + \frac{1}{2}A^2 + \dots + \frac{1}{n!}A^n \right)$.
2. $\|e^A\| \le e^{\|A\|}$ and the e^A is well-defined.
3. $\Phi(t) = e^{tA}$ solve the following initial value problem
$$\dot{\Phi} = A\Phi \qquad \text{and} \qquad \Phi(0) = I.$$

Matrix exponential and stability

$$(1) \qquad \dot{x} = A(t)x + b(t); \quad A \in \mathbb{R}^{n\times n};\ b, x \in \mathbb{R}^n.$$

Matrix exponential:

1. $A = PJP^{-1} \quad \Rightarrow \quad e^A = Pe^J P^{-1}$
2. e^{tA} fundamental matrix; the solution of (1) when $A = \mathrm{konst}$ is
$$x(t) = e^{(t-t_0)A}x(t_0) + \int_{t_0}^{t} e^{(t-s)A}b(s)ds.$$

Stability:

1. Theorem: Φ fundamental matrix, $\dot{\Phi} = A(t)\Phi$.
   - (a) Φ bounded, $t \geq t_0 \Rightarrow$ all solutions of (1) stable
   - (b) $\Phi \to 0$ as $t \to \infty \Rightarrow$ all solutions of (1) asympt. stable
   - (c) Φ unbounded, $t \geq t_0 \Rightarrow$ all solutions of (1) unstable
2. Theorem: $A = \mathrm{konst}$, eigenvalues λ_i.
   - (a) $\max_i \mathrm{Re}\,\lambda_i \leq 0$ and $\lambda_i \neq \lambda_j \Rightarrow$ all solutions of (1) stable
   - (b) $\max_i \mathrm{Re}\,\lambda_i < 0 \Rightarrow$ all solutions of (1) asympt. stable
   - (c) $\max_i \mathrm{Re}\,\lambda_i > 0 \Rightarrow$ all solutions of (1) unstable

Stability when A is non-constant

$$(1) \qquad \dot{x} = A(t)x + b(t); \quad A \in \mathbb{R}^{n\times n};\ b, x \in \mathbb{R}^n.$$

1. Assume: $A(t) = B + C(t)$ where B constant, $\int_{t_0}^{\infty} \|C(s)\| ds < \infty$.

2. Theorem: Φ fundamental matrix, $\dot{\Phi} = B\Phi$.

   (a) Φ bounded, $t \geq t_0 \Rightarrow$ all solutions of (1) stable

   (b) $\Phi \to 0$ as $t \to \infty \Rightarrow$ all solutions of (1) asympt. stable

3. Theorem: B has eigenvalues λ_i.

   (a) $\max_i \operatorname{Re} \lambda_i \leq 0$ and $\lambda_i \neq \lambda_j \Rightarrow$ all solutions of (1) stable

   (b) $\max_i \operatorname{Re} \lambda_i < 0 \Rightarrow$ all solutions of (1) asympt. stable

Lipschitz condition

Definitions: Let $f = (f_1, \ldots, f_n) \in \mathbb{R}^n$.

1. $f(x)$ is Lipschitz in a set $\Omega \subset \mathbb{R}^n$ if there is $L > 0$ such that
$$|f(x) - f(y)| \leq L|x - y| \quad \text{for all} \quad x, y \in \Omega.$$
2. $f(x,t)$ is x-Lipschitz in a set $R \subset \mathbb{R}^{n+1}$ if there is $L > 0$ such that
$$|f(x,t) - f(y,t)| \leq L|x - y| \quad \text{for all} \quad (x,t), (y,t) \in R.$$
3. $f(x,t)$ is locally Lipschitz on Ω if they are Lipschitz on each closed bounded subset $K \subset \Omega$ (or – on each closed bounded ball $\overline{B} \subset \Omega$).
4. Locally x-Lipschitz defined similarly.
5. Lipschitz constant: The smallest L in the above inequalities.
6. $[\sqrt{x_1^2 + x_3^2}, 5, \frac{x_1^2}{1+x_1^2+x_3^2}]$ Lip. in $\mathbb{R}^3$; x^2 only loc. Lip./$\sqrt{x}$ not in $\mathbb{R}$

Lipschitz condition

1. $f(x,t)$ is x-Lipschitz in a set $R \subset \mathbb{R}^{n+1}$ if there is $L > 0$ such that
$$|f(x,t) - f(y,t)| \leq L|x-y| \quad \text{for all} \quad (x,t),(y,t) \in R.$$

2. $f(x)$ is locally x-Lipschitz in R if it is x-Lipschitz on each closed bounded subset $K \subset \Omega$ (or – on each closed ball $\overline{B} \subset \Omega$).

3. Jacobi matrix: $Df(x,t) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$

4. f is C^1 in bounded, convex $R \Rightarrow f$ is x-Lip. in R and $L \leq \max_R \|Df\|$.

Uniqueness and continuous dependence

$$\dot{x}(t) = f(x(t), t), \tag{1}$$

$$x(t_0) = x_0. \tag{2}$$

Uniqueness:

If f is continuous and locally x-Lipschitz in a domain R, then there are not more than one solution of (1) and (2) in R.

Continuous dependence on initial data:

Assume

1. f is continuous and x-Lipschitz in a domain R,
2. x, y solutions of (1) for $t \in (t_0, b)$,
3. $(x(t), t)$, $(y(t), t) \subset R$ for $t \in [t_0, b)$,
4. $\lim_{t \to t_0} x(t) = x_0$ and $\lim_{t \to t_0} y(t) = y_0$.

Then

$$|x(t) - y(t)| \leq e^{L(t-t_0)}|x_0 - y_0| \quad \text{for} \quad t \in [t_0, b).$$

Remark: $x(t) = \phi(t; x_0)$ is continuous in x_0.

Continuous dependence on data

Theorem:

Assume

1. $\dot{x} = f(x,t)$ and $\dot{y} = g(y,t)$ for $t \in (t_0, b)$,
2. $(x(t),t)$, $(y(t),t) \subset R$ for $t \in [t_0, b)$ (R domain),
3. f, g is continuous and f x-Lipschitz in R,
4. $|f(x,t) - g(x,t)| \leq \varepsilon$ ($\varepsilon > 0$) for all $(x,t) \in R$,
5. $\lim_{t\to t_0} x(t) = x_0$ and $\lim_{t\to t_0} y(t) = y_0$.

Then

$$|x(t) - y(t)| \leq e^{2L(t-t_0)}|x_0 - y_0| + \frac{\varepsilon}{L}\sqrt{e^{2L(t-t_0)} - 1}, \quad t \in [t_0, b),$$

where L is the Lipschitz constant of f, i.e. $L \leq \max_R \|Df(x,t)\|$.

Remark: The solution $x(t) = \phi(t; x_0, t_0, f)$ of

$$\dot{x} = f(x,t) \quad \text{and} \quad x(t_0) = x_0$$

is continuous in x_0 and f (and t_0!).

Existence of solutions:

$$\dot{x}(t) = f(x(t), t), \tag{1}$$
$$x(t_0) = x_0. \tag{2}$$

Theorem 1: ("Global" existence)

If f is continuous and x-Lipschitz for $x \in \mathbb{R}^n$ and $|t - t_0| \leq T$, then there exists a solution of (1) and (2) for $|t - t_0| \leq T$.

Theorem 2: (Local existence 1 – Piccard-Lindelöf)

If f is continuous + x-Lipschitz for $|x - x_0| \leq K$, $|t - t_0| \leq T$, then there exists a sol'n of (1) + (2) for $|t - t_0| \leq \min(T, \frac{K}{M})$, $M = \max\limits_{\substack{|t - t_0| \leq T \\ |x - x_0| \leq K}} |f(x,t)|$

Theorem 3: (Local existence 2 – Peano)

If f is continuous for $|x - x_0| \leq K$ and $|t - t_0| \leq T$, then there exists a solution of (1) and (2) for $|t - t_0| \leq \min(T, \frac{K}{M})$ where M as in Thm. 2.

Remark: In Thm's 1 and 2, the solution is unique – but not in Thm 3!

Existence interval and C^1 dependence on x_0.

$$\dot{x}(t) = f(x(t), t), \tag{1}$$

$$x(t_0) = x_0. \tag{2}$$

Theorem 1: f continuous and x-Lipschitz on domain $R \ni (x_0, t_0)$.

(a) There exists a unique solution of (1) and (2) on a maximal existence interval $(t_0 - a, t_0 + b)$ for some $a, b > 0$.

(b) If $b < \infty$, then either $(x(t), t) \to \partial R$ or $|x(t)| \to \infty$ as $t \to b^-$.

Remark:

1. 3 possibilities: $b = \infty$, or x blows up in finite time, or $(x(t), t)$ leaves in finite time the region R of well-posedness.
2. b = existence/life time of the solution of (1) and (2).

Theorem 2: $f \in C^1$ and $\phi(t; x_0)$ (unique) solution of (1) and (2).

Then ϕ is C^1 in x_0 (and t), and $w = \frac{\partial \phi}{\partial x_j}$ is the unique solution of

$$w_t = Df(\phi(t; x_0), t)w, \quad w(0) = e_j.$$

Phasediagram for autonomous systems

(1) $\dot{x} = f(x), \qquad f \in R^n$ independent of t

(2) $x(t_0) = x_0.$

1. Phase trajectory through x_0: $\left\{x(t) : t \in \mathbb{R}, x \text{ solves } (1),\ x(0) = x_0\right\}$

   Tangent (direction) at x: $f(x)$ $[= \dot{x}]$.

   - f continuous $\Rightarrow$ continuity of directions

   Equation: $\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{f_2(x)}{f_1(x)}, \ldots, \frac{dx_n}{dx_1} = \frac{\dot{x}_n}{\dot{x}_1} = \frac{f_n(x)}{f_1(x)}$

   Equilibrium point x_e: $f(x_e) = 0$, and trajectory = one point

2. Phase diagram: Sketch of "all" phase trajectories.

3. Well-posedness for (1) and (2) for all x_0, t

   $\Rightarrow$ trajectories exist + do not cross + pass through every $x_0 \in \mathbb{R}^n$.

4. Separatrix: Trajectory separating regions w. different sol'n behaviour

5. Only in non-linear systems: Many isolated equilibrium points, limit cycles, separatix cycles, and chaos (in $\mathbb{R}^n$, $n \geq 3$).

Linearization

(1) $$\dot{x} = f(x).$$

Linearized about equi. pt. x_0: $f(x) = f(x_0) + Df(x_0)(x - x_0) + \dots$

(2) $$\dot{y} = Df(x_0)y \quad (x - x_0 = y).$$

1. Near x_0, $f(x_0) \neq 0$: Phase diagram $\approx$ straigth lines (if $f \in C^1$).
2. Near x_0, $f(x_0) = 0$ and hyperbolic: $\operatorname{Re} \lambda_i \neq 0$ for eig.val's of $Df(x_0)$
   1. Phasediagram (1) looks like phasediagram (2) near $y = 0$
   2. (1) and (2) has same types of equilibrium points in x_0 and $y = 0$ (including asymptotic lines etc.)
   3. Justification: Hartman-Grobman + Hartman thm's (need $f \in C^2$!).
3. Near x_0, $f(x_0) = 0$ and non-hyperbolic:

   No information from linearization – need higher order theory.

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   No information from linearization – need higher order theory.

Reminder: Stability

$$\dot{x}(t) = f(x(t), t), \tag{1}$$

$$x(t_0) = x_0. \tag{2}$$

Flow ϕ: $x(t) = \phi(t; x_0, t_0)$ solution of (1) and (2).

Stability: Let $x(t) = \phi(t; x_0, t_0)$.

1. $x(t)$ stable for $t \geq t_0$ if for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$|x_1 - x_0| < \delta \quad \Rightarrow \quad |\phi(t; x_1, t_0) - \phi(t; x_0, t_0)| < \varepsilon \quad \text{for all } t > t_0.$$

2. $x(t)$ asymptotic stable for $t \geq t_0$ if stable and there is $\eta > 0$ s.t.

$$|x_1 - x_0| < \eta \quad \Rightarrow \quad |\phi(t; x_1, t_0) - \phi(t; x_0, t_0)| \to 0 \text{ as } t \to \infty.$$

Stability of equilibrium points

Nearly linear systems:

(1) $\dot{x} = Ax + h(x,t); \qquad A \in \mathbb{R}^{n\times n}, \quad x, h \in \mathbb{R}^n.$

Theorem 1:

Assume λ_i eigenvalues of A, $h \in C^1$, and

$$|h(x,t)| = o(|x|) \quad \text{as} \quad x \to 0 \quad \text{uniformly in } t.$$

$\max_i \operatorname{Re}\lambda_i < 0 \;\Rightarrow\; 0$ is an asympt. stable equilibrium pt. of (1).

Autonomous systems: Linearization

(2) $\dot{x} = f(x); \qquad x, f \in \mathbb{R}^n.$

Theorem 2:

Assume $f \in C^1$, $f(x_0) = 0$, and λ_i eigenvalues of $Df(x_0)$.

(a) $\max_i \operatorname{Re}\lambda_i < 0 \;\Rightarrow\; x_0$ is an asympt. stable equilibrium pt. of (2)

(b) $\max_i \operatorname{Re}\lambda_i > 0 \;\Rightarrow\; x_0$ is an unstable equilibrium pt. of (2)

(c) $\max_i \operatorname{Re}\lambda_i = 0 \;\Rightarrow\;$ NO CONCLUSION!!

OBS: Stability based on Df = stability by linearization $\dot{y} = Df(x_0)y$

Liapunov's direct method

- More general method than linearization.
- Need to construct (find) a Liapunov function.

(2) $$\dot{x} = f(x); \qquad x, f \in \mathbb{R}^n.$$

$V(x)$ is a strong Liapunov function for (2) in domain $B \ni 0$ if

1. $V \in C^1(B)$,
2. $V(0) = 0$ and $V(x) > 0$ for $x \in B \setminus \{0\}$,
3. $\dot{V}(0) = 0$ and $\dot{V}(x) < 0$ for $x \in B \setminus \{0\}$, where

$$\dot{V}(x) = \frac{d}{dt} V(x(t)) = \nabla V(x) \cdot \dot{x} = \nabla V(x) \cdot f(x).$$

Theorem 2: Assume $f(0) = 0$ and f Lipschitz in some domain $B \ni 0$.

If $V(x)$ is a strong Liapunov function for (2) in B,
then $x = 0$ is an asymptotically stable equilibrium point for (2).

Liapunov's direct method

(1) $$\dot{x} = f(x); \qquad x, f \in \mathbb{R}^n.$$

Weak (strong) Liapunov function $V(x)$ for (1) in domain $B \ni 0$:

1. $V \in C^1(B)$,
2. $V(0) = 0$ and $V(x) > 0$ for $x \in B \setminus \{0\}$,
3. $\dot{V}(0) = 0$ and $\dot{V}(x) \leq 0$ $(V(x) < 0)$ for $x \in B \setminus \{0\}$, where
4. $\dot{V}(x) = \frac{d}{dt}V(x(t)) = \nabla V(x) \cdot \dot{x} = \nabla V(x) \cdot f(x)$.

Candidates:

$V(x) = \sum_i c_i x_i^2$, or $V(x) = x^T A x$ for $A = A^T$ positive definite, or $V =$ energy/Hamiltonian etc.

Theorem: Assume $f(0) = 0$ and f Lipschitz in some domain $B \ni 0$.

(a) If $V(x)$ is a weak Liapunov function for (1) in B, then $x = 0$ is an stable equilibrium point for (1).

(b) If $V(x)$ is a strong Liapunov function for (1) in B, then $x = 0$ is an asymptotically stable equilibrium point for (1).

Liapunov methods for autonomous systems

(1) $$\dot{x} = f(x); \qquad f, x \in \mathbb{R}^n.$$

Definitions:

1. $\dot{V}(x) = \frac{d}{dt}V(x(t)) = \nabla V(x) \cdot f(x)$
2. $V(x)$ weak (strong) Liapunov function of (1) in domain $B \ni 0$ if
   1. $V \in C^1(B)$,
   2. $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$
   3. $\dot{V}(0) = 0$ and $\dot{V}(x) \leq 0$ ($\dot{V}(x) < 0$) for $x \neq 0$

Theorem: Assume $f(0) = 0$ and f Lipschitz in a domain $B \ni 0$.

1. $V(x)$ weak Lipunov function for (1) $\Rightarrow$ $x(t) \equiv 0$ is stable
2. $V(x)$ strong Lipunov function for (1) $\Rightarrow$ $x(t) \equiv 0$ is asympt. stable
3. $U(x)$ satisfy 1–3 below $\Rightarrow$ $x(t) \equiv 0$ is unstable
   1. $U \in C^1(B)$,
   2. $U(0) = 0$; for any $\delta > 0$, there is $x \in B$ s.t. $|x| < \delta$ and $U(x) > 0$,
   3. there is $\eta > 0$ such that $\dot{U}(x) > 0$ for all x s.t. $|x| < \eta$ and $U(x) > 0$

Examples: $V = \frac{1}{2}(x_1^2 + x_2^2)$, = energy/Hamiltonian; $U = x_1 x_2$, $= x_1^2 - x_2^2$

Invariant domains and domains of attraction

$$(1) \qquad \dot{x} = f(x); \qquad f, x \in \mathbb{R}^n.$$

1. Flow: $\phi(t; x_0)$ solution of (1) and $x(0) = x_0$.
2. $\Omega \subset \mathbb{R}$ invariant (positive invariant) under (1) if $\phi(t; x_0) \in \Omega$ for all $t \in \mathbb{R}$ $(t \geq 0)$ and $x_0 \in \Omega$.

   $\Omega_a = \{x \in \mathbb{R}^n : \lim_{t\to\infty} \phi(t; x) = x_0\}$ domain of attraction of x_0.
3. Lemma: $\Omega = \{x : V(x) \leq c\}$ is positive invariant if $f, V \in C^1$ and
   $$\nabla V(x) \neq 0 \quad \text{and} \quad \dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0 \quad \text{on} \quad V(x) = c.$$
4. Lasalle's invariance principle. Assume:
   1. $f(0) = 0$ and f Lipschitz in $\Omega \ni 0$,
   2. Ω positive invariant, closed and bounded,
   3. $V(x)$ is a weak Liapunov function on Ω,
   4. there is no global in time solution of (1) such that
   $$x(t) \in \Omega \setminus \{0\} \quad \text{and} \quad V(x(t)) = \mathrm{constant} \quad \text{for all} \quad t \in \mathbb{R}.$$

   Then $x = 0$ is asymptotically stable and $\Omega \subset \Omega_a$ (attraction).

Hamiltonian 2×2 systems

$$(1) \qquad \dot{x} = f(x); \qquad x = (x_1, x_2), \quad f = (f_1, f_2).$$

Definition: (1) Hamiltonian if there is a Hamilton function $H(x) \in C^2$ s.t.

$$f_1 = \frac{\partial H}{\partial x_2} \quad \text{and} \quad f_2 = -\frac{\partial H}{\partial x_1}.$$

1. (1) Hamiltonian $\quad \Leftrightarrow \quad \nabla \cdot f = 0 \quad$ (f divergence free)
2. H Hamilton function, $x(t)$ solution of (1) $\Rightarrow$ $H(x(t)) =$ constant
3. Equilibrium points of (1) = critical points of H ($\nabla H = 0$)
4. Classification of equilibrium point x_0 via 2nd derivative test for H:

   $q(x_0) > 0$ center, $q(x_0) < 0$ saddle, $q(x_0) = 0$ no conclusion,

   where $q = \det D^2 H = H_{x_1 x_1} H_{x_2 x_2} - H_{x_1 x_2} H_{x_2 x_1} = \lambda_1(D^2 H) \lambda_2(D^2 H)$.

OBS: Similar results hold for $2n \times 2n$ systems: $f_1 = \nabla_{x_2} H$, $f_2 = -\nabla_{x_1} H$

Index theory for 2×2 systems

$$\dot{x} = f(x); \qquad x, f \in \mathbb{R}^2. \tag{1}$$

1. Polar angle of f: $\phi = \arctan \frac{f_2}{f_1}$
2. Curve Γ: Simple, closed, p.w. C^1, oriented counter cl.wise, $f|_\Gamma \neq 0$
3. Index of Γ: $\boxed{I_\Gamma = \frac{1}{2\pi} \oint_\Gamma d\phi}$
4. $\Gamma = \{y(s) : s_0 \leq s \leq s_T\}$:
   - $I_\Gamma = \dfrac{1}{2\pi} \displaystyle\int_{s_0}^{s_T} \frac{f_1 \frac{d}{ds} f_2 - f_2 \frac{d}{ds} f_1}{f_1^2 + f_2^2} ds;\ \ f_i(s) = f_i(y(s)),$
   - $I_\Gamma = \frac{1}{2\pi} \int_{y(s_0)}^{y(s_T)} \nabla\phi \cdot dy = \frac{\phi(y(s_T)) - \phi(y(s_0))}{2\pi} \in \mathbb{Z}$ since $f(s_0) = f(s_T)$
5. Winding number of $\Gamma_f = \{f(x) : x \in \Gamma\}$: $\frac{1}{2\pi} \int_{\Gamma_f} \frac{X_1 dX_2 - X_2 dX_1}{X_1^2 + X_2^2} = I_\Gamma$.

   $I_\Gamma = \frac{1}{2}(p+q)$ where

   $p =$ number of times $f(x)$ is parallel with any given axis and crosses it counter clockwisely as x traverses Γ counter clockwisely

   $q =$ number of times f cross the same axis clockwisely on Γ.

Index theory for 2×2 systems

$$(1) \qquad \dot{x} = f(x); \qquad x = (x_1, x_2), \; f = (f_1, f_2).$$

Definition:

1. Index of curve Γ: $I_\Gamma = \frac{1}{2\pi} \oint_\Gamma d\phi$
2. Index of equilibrium point x_e:
   $I_{x_e} = I_\Gamma$ for any Γ encircling x_e but no other equilibrium point of (1).

Remarks:

1. $\phi = \arctan \frac{f_2}{f_1}$ polar angle of f.
2. Γ: Simple, closed, p.w. C^1 curve oriented counter cl.wise, $f|_\Gamma \neq 0$.
3. Saddle: $I = -1$; node, spiral, center, closed phase trajectory: $I = 1$
4. There are equilibrium points with $I = n$ for any $n \in \mathbb{Z}$.

Theorem: Assume f is C^2 and $f = 0$ in Ω only at $x_1, \ldots, x_n$. Then

$$I_\Gamma = I_{x_1} + I_{x_2} + \cdots + I_{x_n}$$

for any curve Γ in Ω encircling $x_1, \ldots, x_n$.

Closed trajectories and Poincare sequences

$$\dot{x} = f(x); \quad x, f \in \mathbb{R}^2. \tag{1}$$

Closed trajectories

1. Closed phase trajectories $\Leftrightarrow$ periodic solutions (autonomous syst.)
2. Limit cycles: Isolated closed phase trajectories.
3. Index test: Closed curve Γ surround equilibrium points x_i, index I_i:
   $$\sum I_i \neq 1 \quad \Rightarrow \quad \Gamma \quad \text{is not a phase trajectory.}$$
4. Dulac's test: Ω open, simply connected; $\rho, f = (f_1, f_2) \in C^1(\Omega)$:
   $$\nabla \cdot (\rho f) < 0 \text{ in } \Omega \text{ (or } > 0) \quad \Rightarrow \quad \text{no closed phase trajectory in } \Omega.$$
5. $\rho \equiv 1 \rightarrow$ Bendixon's negative criterion

Poincare sequences:

1. Poincare cross section Σ: curve transversal (=non-parallel) to f
2. Poincare map P_Σ of $x_0 \in \Sigma$: Point of first return of flow $\phi(t; x_0)$ to Σ
3. Poincare sequence: $x_0, P_\Sigma(x_0), \ldots, P_\Sigma^n(x_0), \ldots$ $(P^n = P \circ P \circ \cdots \circ P)$

Poincare Bendixon's theorem

$$(1) \qquad \dot{x} = f(x); \qquad x, f \in \mathbb{R}^2.$$

1. $\Gamma^{\pm}_{x_0} = \{\phi(t, x_0) : t \in [0, \pm\infty)\}$ $\quad$ ($\Gamma_{x_0} = \Gamma^{+}_{x_0} \cup \Gamma^{-}_{x_0}$ trajectory; ϕ flow)
2. $\Omega \subset \mathbb{R}^2$ positive invariant under (1): $\Gamma^{+}_{x_0} \subset \Omega$ for all $x_0 \in \Omega$.
3. ω-limit set of Γ_{x_0}: All z s.t. $x(t_n) = \phi(t_n; x_0) \underset{t_n \to \infty}{\to} z$ for some $\{t_n\}_n$.
4. Cycle: A periodic trajectory but no equi. point (limit or center c.).

Theorem (Poincare-Bendixson):
$\Gamma^{+} \subset$ closed bounded K; no equilibrium points in $\omega_\Gamma \;\Rightarrow\; \omega_\Gamma$ is a cycle.

Lemma 1: $\Gamma^{+} \subset K$, K closed, bounded (=compact)
$\Rightarrow \; \omega_\Gamma \subset K$, $\neq \emptyset$, closed, bounded, connected, invariant under (1).

Lemma 2: Assume L transversal line segment ($L \nparallel f$)

1. Γ crosses L in two different points $\Leftrightarrow$ Γ is not closed.
2. Crossing points of Γ and L are ordered the same way along L as along Γ.

Corollary: $x_0 \in \omega_\Gamma \cap \Gamma \;\Leftrightarrow\; \Gamma$ closed trajectory.

Application of Poincare Bendixon's theorem

$$(1) \qquad \dot{x} = f(x); \qquad x, f \in \mathbb{R}^2.$$

Theorem (Poincare-Bendixson):
$\Gamma^+ \subset$ closed bounded K; no equilibrium points in $\omega_\Gamma \;\Rightarrow\; \omega_\Gamma$ is a cycle.

Cycle: A periodic trajectory but not equilibrium point (limit or center c.)
Corresponds to a periodic solution.

Corollary 1: $K \subset \mathbb{R}^2$ closed, bounded, pos. invariant, with no equi. pt's
$\Rightarrow$ at least one cycle in K.

Lemma: $\Omega = \{x : V(x) \leq c\}$ is positive invariant if $f, V \in C^1$ and

$$\nabla V(x) \neq 0 \quad \text{and} \quad \dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0 \quad \text{on} \quad V(x) = c.$$

Corollary 2: There is at least one cycle in $K = \{x : c_1 \leq V(x) \leq c_2\}$ if

1. K is bounded and contains no equilibrium points,
2. $\nabla V \neq 0$ and $\dot{V} = \nabla V \cdot f \geq 0$ on $V(x) = c_1$,
3. $\nabla V \neq 0$ and $\dot{V} = \nabla V \cdot f \leq 0$ on $V(x) = c_2$.

The Lienard equation

$$(2) \qquad \ddot{x} + f(x,\dot{x})\dot{x} + g(x) = 0 \quad \Leftrightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -f(x_1,x_2)x_2 - g(x_1) \end{cases}$$

Obs:

1. Energy: $E(x_1,x_2) = \frac{1}{2}x_2^2 + G(x_1)$, where $G(x) = \int_0^x g(s)ds$.
2. $\frac{d}{dt}E(x(t)) = \nabla E(x) \cdot \dot{x} = -f(x_1,x_2)x_2^2 \leq 0$ (≥ 0) if $f \geq 0$ ($f \leq 0$)

Theorem (cycles): Equation (2) has at least one cycle if

(a) $f(x_1,x_2)$ continuous, $f < 0$ for $|x| < r$, $f > 0$ for $|x| > R$,

(b) $g(x_1)$ continuous, $g < 0$ for $x_1 < 0$, $g > 0$ for $x_1 > 0$,

(c) $G(x_1) = \int_0^{x_1} g(s)ds \to \infty$ as $|x_1| \to \infty$.

Idea: $x = 0$ only equilibrium point, $E(x) = c_1$ and $E(x) = c_2$ bounds bounded invariant region for c_1 small and c_2 big, use Poincare-Bendixon.

The Lienard equation

$$(1) \qquad \ddot{x} + f(x,\dot{x})\dot{x} + g(x) = 0 \quad \Leftrightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -f(x_1,x_2)x_2 - g(x_1) \end{cases}$$

Energy: $E = \frac{1}{2}x_2^2 + \int_0^x g(s)ds$; $\quad \dot{E} = -f(x_1,x_2)x_2^2$.

Theorem (cycle): Equation (1) has at least one cycle if

(a) $f(x_1,x_2)$ continuous, $f < 0$ for $|x| < r$, $f > 0$ for $|x| > R$,

(b) $g(x_1)$ continuous, $g < 0$ for $x_1 < 0$, $g > 0$ for $x_1 > 0$,

(c) $G(x_1) = \int_0^{x_1} g(s)ds \to \infty$ as $|x_1| \to \infty$.

Theorem (centre): Equation (1) has a centre at $x = 0$ if for $|x| < R$:

(a) $f = f(x_1)$ continuous, odd, one sign for $x_1 < 0$,

(b) $g(x_1)$ continuous, odd, $g > 0$ for $x_1 > 0$,

(c) $g(x_1) > \alpha f(x_1) \int_0^{x_1} f(s)ds$ for $\alpha > 1$.

Obs: Orbits loose energy in $\{x_1 \geq 0\}$ but gain same amount in $\{x_1 < 0\}$.

Theorem (limit cycle): Equation (1) has one and only one cycle if

(a) $F(x) = \int_0^x f(s)ds$, g locally Lipschitz,

(b) g odd, $g > 0$ for $x > 0$,

(c) F odd, $= 0$ only at $x = 0, \pm a$, $F \underset{x\to\infty}{\to} \infty$, monotonically for $x > a$.

Obs: The proof uses the Lienard plane: $\dot{x}_1 = x_2 - F(x_1)$, $\dot{x}_2 = -g(x_1)$