Diff. likn. 16.2.12

A. Lipschitz betingelse

$f = (f_1, \ldots, f_n)$, $f = f(x,t)$, $x \in \mathbb{R}^n, t \in \mathbb{R}$

Def. 1: $f(x,t)$ x-Lipschitz i $R \subset \mathbb{R}^{n+1}$ hvis det fins $L \geq 0$ s.a.

$$|f(x,t) - f(y,t)| \leq L|x-y| \text{ for alle } (x,t), (y,t) \in R.$$

Def. 2:

a) $f \in C^1(R;\mathbb{R}^n)$, $R \subset \mathbb{R}^{n+1}$ åpen, hvis

$\frac{\partial f_i}{\partial x_j}$, $\frac{\partial f_i}{\partial t}$, $i,j = 1,\ldots,n$, eksisterer og er kont. i R.

b) Jacobimatrise til $f(x,t)$:

$$D_x f = Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Eks. 1:

$$f(x_1, x_2, t) = \begin{bmatrix} t^2 \sin x_1 \\ e^{x_1 - x_2} + \sin t \end{bmatrix}$$

$$Df = \begin{bmatrix} t^2 \cos x_1 & 0 \\ e^{x_1 - x_2} & -e^{x_1 - x_2} \end{bmatrix}, \quad \frac{\partial f}{\partial t} = \begin{bmatrix} 2t \sin x_1 \\ \cos t \end{bmatrix}$$

$f, Df, \frac{\partial f}{\partial t}$ kont. for alle t, x_1, x_2

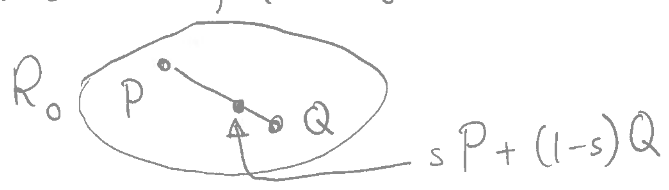
$\Longrightarrow f \in C^1(\mathbb{R}^3; \mathbb{R}^2)$.

Lem. 1: Hvis $f \in C^1(R; \mathbb{R}^n)$, $R \subset \mathbb{R}^{n+1}$ domene, og $R_0 \subset R$ lukka, begrenset, konveks, da er f x-Lip. på R_0 og

$$L_{f, R_0} \leq \max_{(x,t) \in R_0} \|Df(x,t)\| \leq n \max_{i,j} \max_{(x,t) \in R_0} \left| \frac{\partial f_i}{\partial x_j}(x,t) \right| < \infty.$$

Rem. 1:

a) R_0 konveks: $P, Q \in R_0 \Rightarrow sP + (1-s)Q \in R_0$, $s \in [0,1]$



b) Alle baller er konvekse (og begrenset).

Eks. 1: (forts.)

$$\|Df(x,t)\| \overset{\text{sjk.}}{\leq} 2\max\left(t^2|\cos x_1|,\ e^{x_1-x_2}\right)$$

$$\leq 2\max\left(\max(a^2,b^2),\ e^{2(|x_0|+r)}\right) < \infty$$

for alle $x \in \underbrace{\{x : |x-x_0| \leq r\}}_{\overline{B}(x_0,r)}$, $t \in [a,b]$.

Lem. 1 $\Rightarrow$ f x-Lip. på $\overline{B}(x_0,r) \times [a,b]$

for alle $x_0 \in \mathbb{R}^2$, $r>0$, $a<b$

$\Rightarrow$ f lok. x-Lip i $\mathbb{R}^3$

Kor. 1: $f \in C^1(R;\mathbb{R}^n)$, $R \subset \mathbb{R}^{n+1}$ domene

$\Rightarrow$ f lok. x-Lip. i R.

[Lem. 1 $\Rightarrow$ f x-Lip. på alle lukka baller i R

$\Leftrightarrow$ f lok. x-Lip. i R]

Eks. 2:

$f, x \in \mathbb{R}^1 \Rightarrow |f(x)-f(y)| = \left|\int_y^x f'(s)\,ds\right| \leq \max\limits_{s\in[y,x]} |f'(s)| \cdot |x-y|$

Bevis Lem. 1:

1.) $g(s) = f_i(h(s))$, $h(s) = sx-(1-s)y$, $s \in \mathbb{R}$

g kont., $g(1) = f_i(x,t)$, $g(0) = f_i(y,t)$

2.) $f_i(x,t) - f_i(y,t) = g(1) - g(0)$

$$= \int_0^1 \frac{d}{ds} g(s)\, ds$$

$$\overset{\text{kj. regel}}{=} \int_0^1 \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(h(s)) \cdot \underbrace{\frac{dh_j}{ds}}_{\overset{\shortparallel}{(x_j - y_j)}} ds$$

3.) Obs: $h(s) \in R_0$ (konveks)

2) og def. av Df gir da

$$f(x,t) - f(y,t) = \int_0^1 Df(h(s)) \cdot (x - y)\, ds$$

$$\Rightarrow |f(x,t) - f(y,t)| \leq \max_{s \in [0,1]} \|Df(h(s))\| \cdot |x-y|$$

$$\leq \max_{(x,t) \in R_0} \|Df(x,t)\| \cdot |x-y|$$

$$\leq n \cdot \max_{i,j} \max_{(x,t) \in R_0} \left| \frac{\partial f_i}{\partial x_j}(x,t) \right| \cdot |x-y|$$

Dvs. f x-Lip. i R_0 og $L_{f,R_0} \leq \max_{(x,t) \in R_0} \|Df(x,t)\|$ $\square$

B. Entydighet

(1) $\dot{x} = f(x,t)$

(2) $x(t_0) = x_0$

der $f = (f_1, \ldots, f_n) \in \mathbb{R}^n$, $x \in \mathbb{R}^n$.

Anta $x(t), y(t)$ to løsn. av (1), og

$$\sigma(t) := |x(t) - y(t)|^2$$

Obs:

$$\dot{\sigma}(t) = 2(x-y)(\dot{x}-\dot{y}) \overset{(1)}{=} 2(x-y)\big(f(x,t) - f(y,t)\big)$$

Hvis f kont. + x–Lip.:

$$\dot{\sigma} \overset{C-S}{\leq} 2|x-y| \cdot |f(x,t) - f(y,t)|$$

$$\overset{x\text{-Lip.}}{\leq} 2|x-y| L |x-y| = 2L\sigma$$

Int. faktor e^{-2Lt}:

$$\frac{d}{dt}\left(e^{-2Lt}\sigma(t)\right) = e^{-2Lt}(\dot{\sigma} - 2L\sigma) \overset{\dot{\sigma} \leq 2L\sigma}{\leq} 0$$

Int. $\int_{t_0}^{t}$

$$e^{-2Lt}\sigma(t) - e^{-2Lt_0}\sigma(t_0) \leq 0$$

(*) $\quad \Rightarrow \quad \sigma(t) \leq e^{2L(t-t_0)}\sigma(t_0)$

Vi har da:

Tm. 1:

Anta:

i) f kont. + x-Lip. i domene $R \subset \mathbb{R}^{n+1}$

ii) $x(t), y(t)$ løser (1) for $t \in (t_0, b)$

iii) $(x(t), t), (y(t), t) \in R$ for $t \in [t_0, b)$

iv) $\lim_{t \to t_0^+} x(t) = x_0$, $\lim_{t \to t_0^+} y(t) = y_0$

Da er

$$|x(t) - y(t)| \leq e^{L|t-t_0|} |x_0 - y_0| \quad \text{for } t \in [t_0, b).$$

$$[\,(*) \Rightarrow \sqrt{\sigma(t)} \leq e^{L(t-t_0)} \sqrt{\sigma(t_0)}, \quad \sqrt{\sigma} = |x-y|\,]$$

Tm. 2: (entydighet)

Hvis f er kont. og x-Lip. i domene $R \subset \mathbb{R}^{n+1}$, da fins det ikke mer enn én løsn. av init. verdi prob. (1) og (2) i R.

Bevis:

Anta to løsn. x, y i R, dvs.

$$(x(t), t), (y(t), t) \in R \text{ for } t \in (a, b), \quad a < t_0 < b.$$

$$\overset{\text{Tm. 1}}{\Longrightarrow} |x(t)-y(t)| \le e^{L|t-t_0|}|x(t_0)-y(t_0)| \overset{(2)}{=} 0, \quad t\in[t_0,b)$$

<u>$t<t_0$</u>: x løser (1) for $t<t_0 \Rightarrow \tilde{x}(t)=x(-t)$ løser

$$\dot{\tilde{x}}(t) = -\dot{x}(-t) \overset{(1)}{=} -f(x(-t),-t), \quad t>-t_0$$

Obs: $(\tilde{x}(t),t), (\tilde{y}(t),t) \in R$ for $t\in(-b,-a)$

$$\Rightarrow |x(-t)-y(-t)| = |\tilde{x}(t)-\tilde{y}(t)|$$
$$\overset{\text{Tm. 1}}{\le} e^{L|t-(-t_0)|}|\tilde{x}(-t_0)-\tilde{y}(-t_0)| \overset{(2)}{=} 0, \quad t\in[-t_0,-a).$$

Konklusjon: $x(t)=y(t)$, $t\in(a,b)$ $\square$

<u>Kor. 2:</u>

f kont. og lok. x-Lip. i domene $R\subset\mathbb{R}^{n+1}$

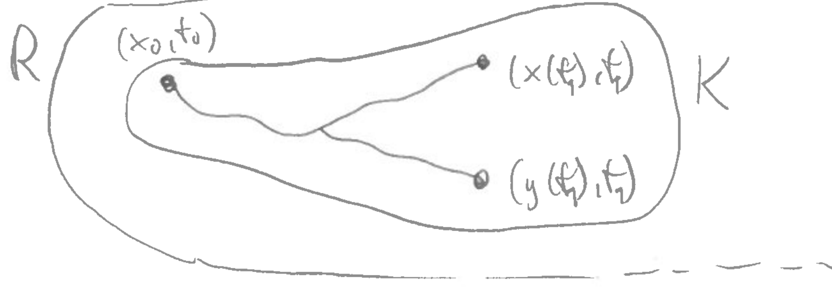
$\Rightarrow$ det fins ikke mer enn én løsn. av (1) og (2) i R.

<u>Bevis:</u>

Anta to løsn. x, y og t_1 s.a. $x(t_1)\neq x(t_1)$.

Da fins lukka, begrenset $K\subset R$ s.a.

$$(x(s),s), (y(s),s) \in K \text{ for } s\in[t_0,t_1]$$



Men f x–Lip. i K, så $x \equiv y$ i K

v. Tm. 2. Dvs. $x(t_1) = y(t_1)$ og motsigelse,

dvs fins ingen t_1 s.a. $x(t_1) \neq y(t_1)$ i $\mathbb{R}$. $\square$

Eks. 3:

$$(**) \quad \dot{x} = f(x,t), \quad f(x,t) = \begin{bmatrix} t^2 \sin x \\ e^{x_1 - x_2} + \sin t \end{bmatrix}$$

Eks. 1 $\Rightarrow$ f lok. x–Lip. i $\mathbb{R}^{n+1}$ (C^1)

$\overset{\text{Kor. 2}}{\Longrightarrow}$ $(**)$ og (2) har ikke mer enn én løsn. for alle $t \in \mathbb{R}$.

Rem. 2:

a) f lok. x–Lip. $\overset{\text{Tm. 1}}{\Longrightarrow}$ $x(t) = \varphi(t; x_0)$ avh. kont. av x_0 (init. data)

b) $\dot{x} = 3\sqrt[\frac{3}{2}]{x}$, $x(0) = 0$: Løsn. $x = 0$ og $x = t^3$

$\sqrt[\frac{3}{2}]{x}$ ikke $(x-)$ Lip. i $x = 0$ (sjk.)

c) Entydighet for $t > t_0$ (men ikke for $t < t_0$) også m. ensidig x-Lip. bet. (Øv. 6):

$$(f(x,t) - f(y,t)) \cdot (x-y) \leq L|x-y|^2, \quad t \geq t_0, \; x, y \in \mathbb{R}^n$$

d) Mer generelle resultater fins. F. eks.

$$f(x) = \begin{cases} |x| - |x| \ln|x|, & |x| < 1 \\ 1, & |x| \geq 1 \end{cases} \quad \text{ikke Lip.}$$

men $\dot{x} = f(x)$ har entydig løsn. for $t \in \mathbb{R}$ (Øv. 6)