Diff.-likn.
22.3.2012

1.)

Neste år: Ta fordløg: 1./2. time?
evt. e. disk. av førediagr.

Beskjeder

A. 2×2 Hamiltonske syst.

(1) $\quad \dot{x} = f(x)$

der $x = (x_1, x_2)$, $f = (f_1, f_2) \in C^1$.

Def. 1:

a) (1) Hamiltonsk syst. hvis fins $H(x) \in C^2$

s.a. $\quad f_1 = \dfrac{\partial H}{\partial x_2}$ og $f_2 = -\dfrac{\partial H}{\partial x_1}$

b) H = Hamilton funk. for (1)

Lem. 1:

(1) Hamiltonsk $\Leftrightarrow$ $\dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2} \equiv 0$ $(\operatorname{div} f \equiv 0)$

[$\Rightarrow$) $\dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2} = \dfrac{\partial}{\partial x_1}\dfrac{\partial H}{\partial x_2} + \dfrac{\partial}{\partial x_2}\left(-\dfrac{\partial H}{\partial x_1}\right) = 0$

$\Leftarrow$) $H = \int_0^{x_2} f_1(x_1, s)\,ds - \int_0^{x_1} f_2(s, 0)\,ds$ Ham. funk. når $\dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2} = 0$ (sjekk!)

$\dfrac{\partial H}{\partial x_1} = \int_0^{x_2} \dfrac{\partial f_1}{\partial x_1} - f_2(x_1, 0) = -\underbrace{\int_0^{x_2} \dfrac{\partial f_2}{\partial x_2} - f_2(x_1,0)}_{= -f_2(x_1,x_2)}$

Obs. 1:

i) H Ham. funk $\Rightarrow$ H + konst. Ham. funk.

ii) $\int_0^{x_2} f_1(x_1, s)\,ds - \int_0^{x_1} f_2(s, 0)\,ds$ Ham. funk. for (1)

iii) (1) Ham. $\Leftrightarrow$ f div.-fritt $(\nabla \cdot f = 0)$

Eks. 1:

$$(*) \begin{cases} \dot{x}_1 = x_2(13 - x_1^2 - x_2^2) = f_1 \\ \dot{x}_2 = 12 - x_1(13 - x_1^2 - x_2^2) = f_2 \end{cases}$$

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_2(-2x_1) + (-x_1)(-2x_2) = 0$$

$\Rightarrow$ $(*)$ Ham.

$$\frac{\partial H}{\partial x_2} = f_1 \Rightarrow H = \int_0^{x_2} f_1(x_1, s)\,ds + C(x_1)$$

$$= \frac{13}{2}x_2^2 - \frac{1}{2}x_1^2 x_2^2 - \frac{1}{4}x_2^4 + C(x_1)$$

$$f_2 = -\frac{\partial H}{\partial x_1} = x_1 x_2^2 - C'(x_1)$$

$$\overset{(*)}{\Longrightarrow} C'(x_1) = -12 + x_1(13 - x_1^2)$$

$$\Rightarrow C(x_1) = -12x_1 + \frac{13}{2}x_1^2 - \frac{1}{4}x_1^4 + C$$

La $C = 0$,

$$H = \int_0^{x_2} f_1 + C(x_1)$$

$$= -12x_1 + \frac{13}{2}(x_1^2 + x_2^2) - \frac{1}{4}(x_1^4 + x_2^4) - \frac{1}{2}x_1^2 x_2^2$$

Lem. 2:

H Ham. funk. for (1) og x(t) løser (1)

$\Rightarrow$ $H(x(t))$ = konst.

$$\left[\frac{d}{dt} H(x(t)) = \frac{\partial H}{\partial x_1}\dot{x}_1 + \frac{\partial H}{\partial x_2}\dot{x}_2 \overset{(1)\ \text{Ham.}}{=} 0 \right]$$

Obs. 2:

i) $H(x(t))$ er bevart.

ii) Nivåkurver $H(x_1, x_2)$ er fasebaner til (1)

iii) x_0 likev. pkt. for (1)

$$\Updownarrow$$

$$\frac{\partial H}{\partial x_1}(x_0) = -f_2(x_0) = 0, \quad \frac{\partial H}{\partial x_2}(x_0) = f_1(x_0) = 0$$

$$\Updownarrow$$

x_0 krit. pkt. for H

iv) 2. deriv. testen:

$$q = \det D^2H = \det \begin{bmatrix} H_{x_1x_1} & H_{x_1x_2} \\ H_{x_2x_1} & H_{x_2x_2} \end{bmatrix}$$

$$= H_{x_1x_1}H_{x_2x_2} - H_{x_1x_2}H_{x_2x_1}$$

$$= \lambda_1(D^2H) \cdot \lambda_2(D^2H)$$ [λ_i krumning i r_i retn. $\rightarrow$ x_0 min $\Rightarrow$ $\lambda_1, \lambda_2 > 0$]

$q(x_0) > 0$ x_0 strengt min. el. max. for H

$q(x_0) < 0$ x_0 sadel for H

$q(x_0) = 0 \Rightarrow$ ingen konklusjon

v) Nær str. min./max. x_0 for H

→ Nivåkurver (= fasebaner) lukka kurver

→ x_0 senter for (1)

vi) Sadel for $H \Rightarrow$ sadel for (1) (linearisering!)

Vi har vist.

Lem. 3:

Anta (1) Ham. syst. og $f(x_0) = 0$.

a) $q(x_0) > 0 \Rightarrow x_0$ senter

b) $q(x_0) < 0 \Rightarrow x_0$ sadel

c) $q(x_0) = 0 \Rightarrow$ ingen konklusjon

(høgere ordens likev. pkt.)

Eks. 1: (forts.)

Likev. pkt. for (*): (sjk.)

$(1,0), (3,0), (-4,0)$ [x_2 må vere 0!]

$$q = \det D^2H = H_{x_1x_1}H_{x_2x_2} - H_{x_1x_2}H_{x_2x_1}$$

$$\overset{(*)\text{ Ham.}}{=} \left(-\frac{\partial f_2}{\partial x_1}\right)\frac{\partial f_1}{\partial x_2} - \left(-\frac{\partial f_2}{\partial x_2}\right)\frac{\partial f_1}{\partial x_1} \qquad \left[f_1 = \frac{\partial H}{\partial x_2},\ f_2 = -\frac{\partial H}{\partial x_1}\right]$$

$$\overset{\text{sjk.}}{=} (13 - 3x_1^2 - x_2^2)\cdot(13 - x_1^2 - 3x_2^2) - 4x_1^2x_2^2$$

	$(1,0)$	$(3,0)$	$(-4,0)$
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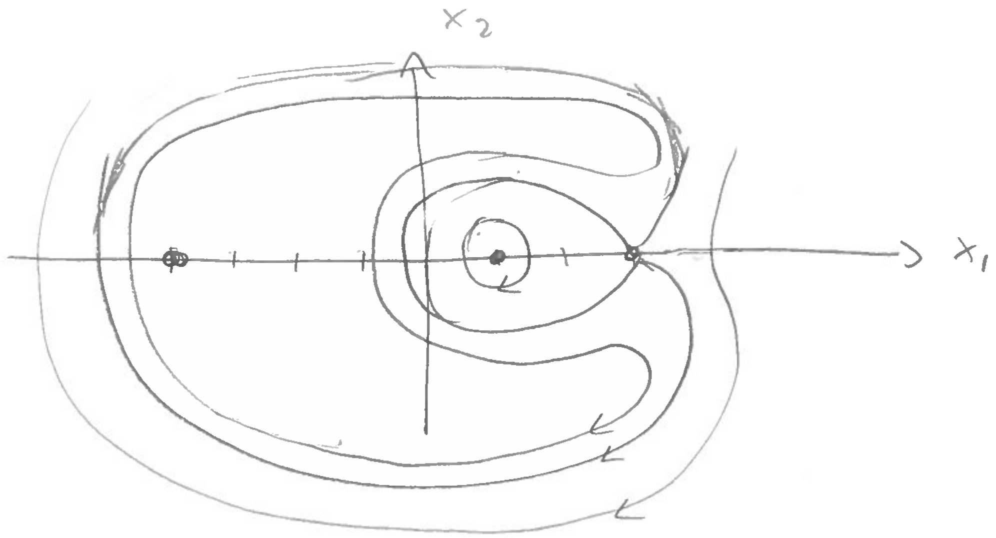


Obs. 3:

i) Lem 3. kan påvise senter (lin. funker ikke)

ii) Konklusjon Lem 3 = konkl. for linearisert syst. (sjk!)

iii) Ham. syst. har ingen noder/spiraler!

(gir ingen bevaring!)

iv) Alt over gjelder $2n \times 2n$ Ham. syst:

$$\dot{x}_1 = +\nabla_{x_2} H(x_1, x_2), \quad \dot{x}_2 = -\nabla_{x_1} H(x_1, x_2)$$

<u>Men</u>: Lem 3: senter når alle eg.v. til $D^2H < 0$ (el. >0) etc.

v) Konservative syst. er Hamiltonske:

$$\ddot{x} = -\nabla\varphi(x) \iff \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\nabla\varphi(x_1) \end{cases} \Rightarrow H(x_1, x_2) = \tfrac{1}{2}x_2^2 + \varphi(x_1)$$

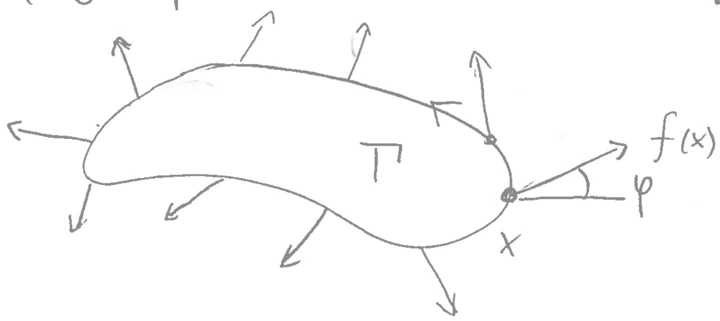
Ham. funk = total energi!

B. Indeks teori

Anta:

1.) Γ enkel, lukka, stk.-vis C^1 kurve, orientert mot kl.

2.) $f(x) \neq 0$ for $x \in \Gamma$ (ikke likev.-pkt. her)



Def. 2:

a) Polarvinkel til f i x: $\varphi(x) = \arctan \frac{f_2(x)}{f_1(x)}$

b) Indeks til Γ: $\boxed{I_\Gamma = \frac{1}{2\pi} \oint_\Gamma d\varphi}$

Rem. 1:

i) Γ C^1 $\Rightarrow$ fins C^1 param. $y(s)$ s.a. $\dot{y} \neq 0$

ii) $y(s)$, $s \in [s_0, s_T]$, C^1 param. for Γ

$$\Rightarrow \oint_\Gamma d\varphi = \oint_\Gamma \left(\frac{d}{ds} (\varphi(x(s))) \right) ds =$$

$$\overset{\text{sjk.}}{=} \int_{s_0}^{s_1} \frac{f_1(y(s)) \frac{d}{ds} f_2(y(s)) - f_2(y(s)) \frac{d}{ds} f_1(y(s))}{f_1^2(y(s)) + f_2^2(y(s))} ds$$

Eks. 2:

$$f(x) = \begin{bmatrix} 2x_1^2 - 1 \\ 2x_1 x_2 \end{bmatrix}$$

Γ: $y(s) = (\cos s, \sin s)$, $s \in [0, 2\pi]$ (sirkel)

Sjk.: $f(y(s)) = \begin{pmatrix} \cos 2s \\ \sin 2s \end{pmatrix}$

$$I_\Gamma = \frac{1}{2\pi}\int_0^{2\pi} \frac{f_1 \frac{d}{ds} f_2 - f_2 \frac{d}{ds} f_1}{f_1^2 + f_2^2}\, ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(2s)2\cos 2s - \sin 2s\,(-2\overset{\sin}{\cancel{\cos}}\,2s)}{1}\, ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2\, ds = \underline{\underline{2}}$$

vindingstall

Obs. 4:

i) Γ fra x_0 til x_1 $\Rightarrow \int_\Gamma d\varphi = \int_{x_0}^{x_1} \nabla\varphi \cdot dx = \varphi(x_1) - \varphi(x_0)$

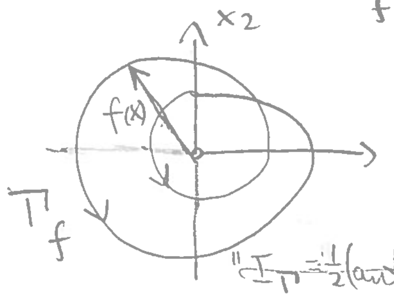
ii) Γ lukka $\Rightarrow \underset{\text{start}}{f(y(s_0))} = \underset{\text{stopp}}{f(y(s_T))}$

$$\Rightarrow \varphi(y(s_0)) = \varphi(y(s_T)) + n\cdot 2\pi\ ;\ n \in \mathbb{Z}$$

$$\overset{i)}{\Rightarrow} I_\Gamma = \frac{1}{2\pi}\big(\varphi(y(s_T)) - \varphi(y(s_0))\big) = n \in \mathbb{Z}$$

iii) I_Γ = vindingstall for $\Gamma_f = \{f(x) : x \in \Gamma\}$

$$I_\Gamma = \frac{1}{2\pi}\oint_{\Gamma_f} \underbrace{\frac{X_1 dX_2 - X_2 dX_1}{X_1^2 + X_2^2}}_{= d\left(\arctan \frac{X_2}{X_1}\right)}\ ;\ X_1 = f_1(x),\ X_2 = f_2(x)$$



$\left(I_{\Gamma_f} = 2,\ \begin{array}{l} 2 \text{ runder mot kl.} \\ 0 \text{ ——"—— m. kl.} \end{array}\right)$

"~~$I_\Gamma = \frac{1}{2}$(antall~~ runder mot kl. − ant. runder m. kl.)"

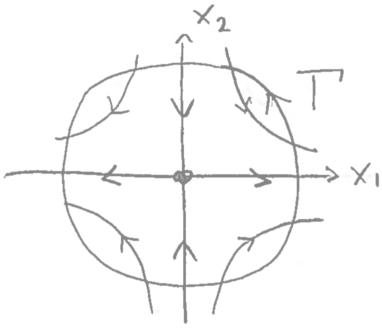
Dvs.:

Lem. 4: $I_\Gamma = \frac{1}{2}(p - q)$ der

p = ant. ganger f krysser en gitt (vilk.) akse mot kl. under ett omløp av Γ

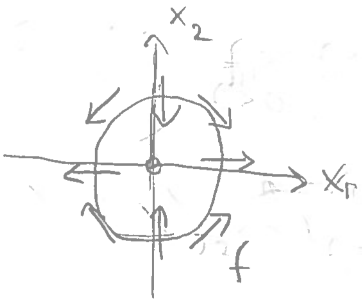
q = ——— " ——— samme akse m. kl.

Eks. 3: Sadel



$$\begin{cases} \dot{x}_1 = +x_1 \\ \dot{x}_2 = -x_2 \end{cases}$$

$\Gamma : x^2 + y^2 = 1$



x_1 – aksen krysses

$2\times$ m. kl.

$0\times$ mot kl.

$\Rightarrow I_\Gamma = -1$