Diff. likn. 22.3.2012

A. Indeksteori

(I) $\dot{x} = f(x)$

der $x = (x_1, x_2)$, $f = (f_1, f_2) \in C^1$.

Begreper:

i) $\varphi(x) = \arctan \frac{f_2(x)}{f_1(x)}$ (polarvinkel)

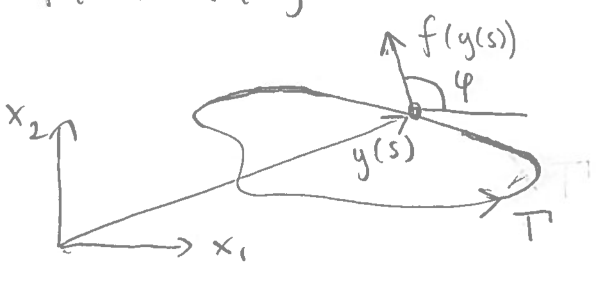
ii) $\Gamma = \{y(s) : s \in [s_0, s_T]\}$ lukka, mot kl., $f \neq 0$ på Γ

iii) $\Gamma_f = \{f(x) : x \in \Gamma\}$

iv) Indeks til Γ:

$$\longleftarrow \quad I_\Gamma = \frac{1}{2\pi}\int_\Gamma d\varphi = \underbrace{\frac{1}{2\pi}\int_{\Gamma_f} \frac{x_1\,dx_2 - x_2\,dx_1}{x_1^2 + x_2^2}}_{\text{vindingstall } \Gamma_f} = \frac{1}{2\pi}\int_{s_0}^{s_T} \frac{f_1 \frac{d}{ds} f_2 - f_2 \frac{d}{ds} f_1}{f_1^2 + f_2^2}\,ds$$

der $f_i(s) = f_i(y(s))$.



Lem. 1:

Anta:

(A1) Γ enkel lukka stk.-vis C^1 kurve, mot kl.

og $f(x) \neq 0$ for $x \in \Gamma$

(A2) $\Omega \subset \mathbb{R}^2$ enkel smh. (ingen hull), åpen, $\Gamma \subset \Omega$.

(A3) $f \in C^2$ og $f(x) \neq 0$ for $x \in \Omega$.

Da er $I_\Gamma = 0$.

Bevis:

$$2\pi I_\Gamma = \oint_\Gamma d\varphi \overset{\text{kj. regel}}{=} \oint_\Gamma \frac{\partial \varphi}{\partial x_1} dx_1 + \frac{\partial \varphi}{\partial x_2} dx_2$$

$$\overset{\text{Greens tm.}}{=} \iint_{D_\Gamma} \underbrace{\frac{\partial}{\partial x_1}\left(\frac{\partial \varphi}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial \varphi}{\partial x_1}\right)}_{=0 \quad \varphi_{x_1 x_2} = \varphi_{x_2 x_1}} dx_1 dx_2 = 0$$

der $D_\Gamma \subset \Omega$ er s.a. $\partial D_\Gamma = \Gamma$ (randen)

Greens ok pga (A1), (A2) og siden $D\varphi$ er C^1 i Ω

v. (A3). [sjk.: bruk at $f \in C^2$ og $f \neq 0$] $\square$

Korollar:

Anta:

1.) Γ, Γ' to kurver som oppfyller (A1)

2.) Γ' ligger innenfor Γ

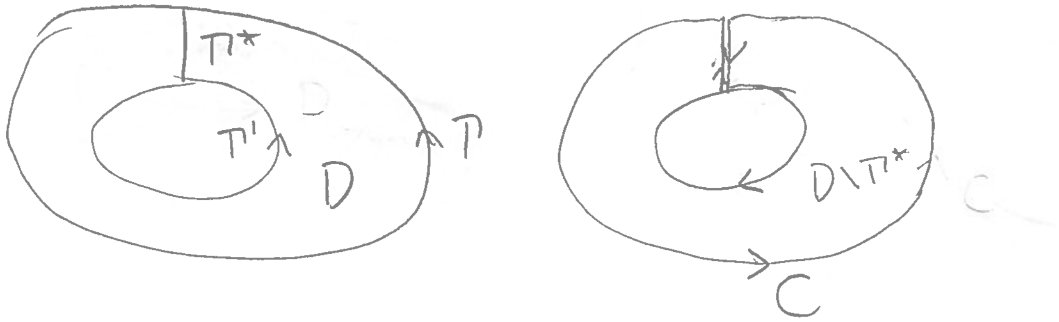
3.) $f \in C^2$ og $f \neq 0$ i området D mellom Γ og Γ'

Da er $I_\Gamma = I_{\Gamma'}$

Bevis:

La $\Gamma^* =$ linjestk. fra Γ til Γ'

og $C = \partial(D \setminus \Gamma^*)$, randen til $D \setminus \Gamma^*$ (mot kl.)



Obs: $C = \Gamma \cup (-\Gamma') \cup \Gamma^* \cup (-\Gamma^*)$

$$0 \overset{\text{Lem. 1}}{=} 2\pi I_C = \oint_C d\varphi = \left(\int_\Gamma + \int_{-\Gamma'} + \int_{\Gamma^*} + \int_{-\Gamma^*} \right) d\varphi$$

$$= \oint_\Gamma d\varphi - \oint_{\Gamma'} d\varphi + \underbrace{\int_{\Gamma^*} d\varphi - \int_{\Gamma^*} d\varphi}_{=0}$$

$$= 2\pi I_\Gamma - 2\pi I_{\Gamma'} \qquad \square$$

Obs. 1:

i) Generelt:

$I_{\Gamma_1} = I_{\Gamma_2}$ hvis $f \neq 0 + C^2$ i $D_1 \cup D_2 \cup D_3 \cup D_4$

[Bruk. Lem. 1 på hver D_i]

ii) x_0 eneste likev. pkt. i $\Omega \subset \mathbb{R}^2$ (og f C^2)

$\Rightarrow I_\Gamma = I_{\Gamma'}$ for alle kurver $\Gamma, \Gamma' \subset \Omega$

som oppfyller (A1) og omslutter x_0.

<u>Def. 1:</u> <u>Indeks</u> til isolert likev. pkt. x_0,

$$I_{x_0} = I_\Gamma$$

for en vilk. Γ om x_0 som ikke omslutter andre likev. pkt. [(A1), (A2) må holde og $f \in C^2$].

<u>Obs. 2:</u>

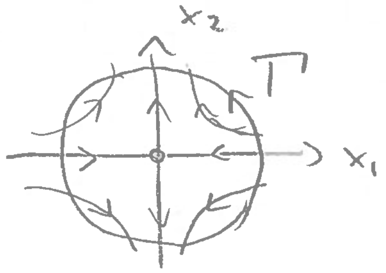
i) x_0 isolert likev. pkt.

$\Rightarrow$ fins $\varepsilon > 0$ s.a. $f(x) \neq 0$ for $0 < |x - x_0| < \varepsilon$

ii) I_{x_0} veldef. v. Obs. 1

<u>Eks. 1:</u> Sadel

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = x_2 \end{cases}$$



Sadel i $x = 0$

Sirkel $\Gamma = \{(\cos s, \sin s);\ s \in [0, 2\pi)\}$

$$I_0 \overset{\text{Def. 1}}{=} I_\Gamma = \frac{1}{2\pi}\int_0^{2\pi} \frac{f_1 \frac{d}{ds} f_2 - f_2 \frac{d}{ds} f_1}{f_1^2 + f_2^2}\, ds$$

der $f(s) = f(\cos s, \sin s) = \begin{bmatrix} -\cos s \\ \sin s \end{bmatrix}$, $\frac{d}{ds} f = \begin{bmatrix} \sin s \\ \cos s \end{bmatrix}$.

Dvs.
$$I_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{-\cos^2 s - \sin^2 s}{\cos^2 s + \sin^2 s}\, ds = -1$$

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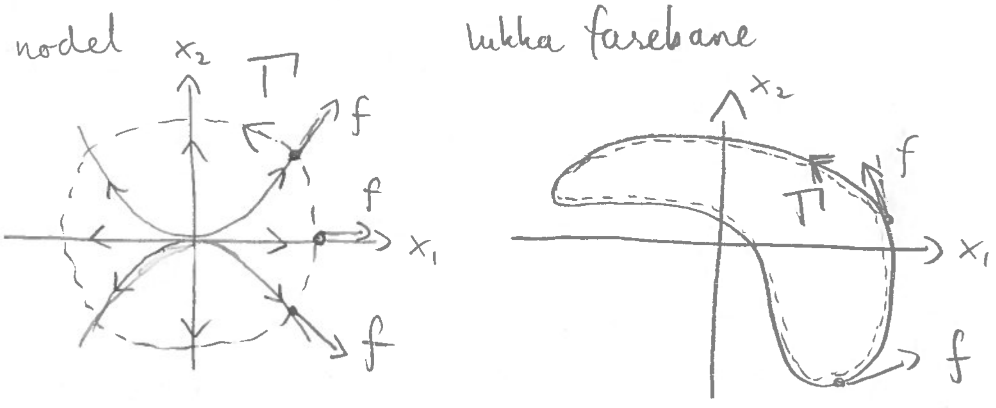
Obs. 3:

i) Sadel : $I = -1$

ii) Node, spiral, senter : $I = +1$

iii) Γ = lukka fasebane : $I_\Gamma = 1$



f krysser x_1-aksen $2\times$ mot kl. ($0\times$ m. kl.)

$\underset{\text{sirk}}{\Longrightarrow}$ $I = I_\Gamma = \frac{1}{2}(2-0) = \underline{\underline{1}}$

Tm. 1:

Anta:

1.) Γ, Ω oppfyller (A1), (A2)

2.) f er C^2 i Ω

3.) $f \neq 0$ i Ω utenom $x_1, \dots, x_n$ ($f(x_i) = 0$)

Hvis Γ omslutter $x_1, \dots, x_n$, da er

$$I_\Gamma = \sum I_i$$

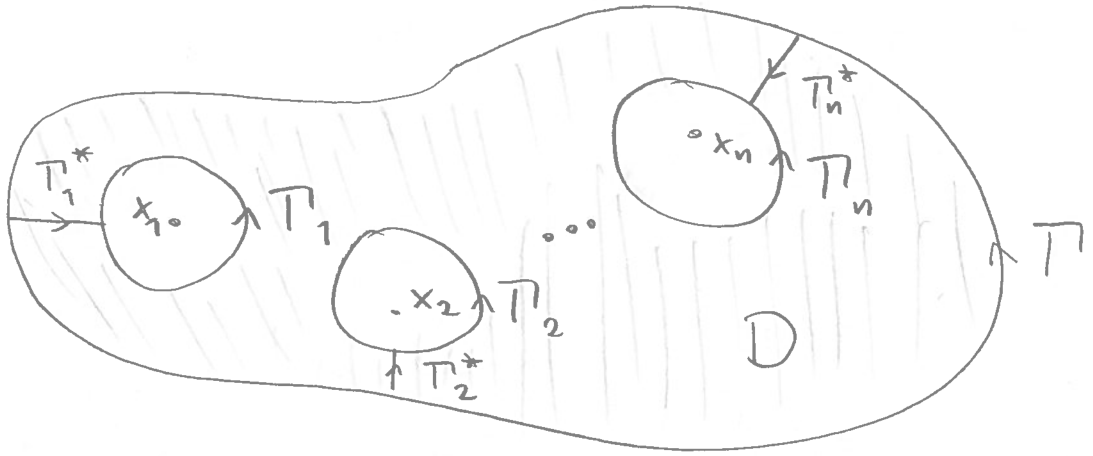
der I_i er indeks til likev. pkt. x_i.

Bevis:

Kurve Γ_i omslutter kun x_i

— " — Γ_i^* linje fra Γ_i til Γ

krysser ingen andre Γ_i el. Γ_i^*



D = område utenfor Γ_i og innenfor Γ

$$C = \partial(D \setminus (\Gamma_1^* \cup \dots \cup \Gamma_n^*)) \quad \text{(mot. kl.)}$$

$$= \Gamma \cup (-\Gamma_1) \cup \dots \cup (-\Gamma_n)$$

$$\cup\, \Gamma_1^* \cup \dots \cup \Gamma_n^*$$

$$\cup\, (-\Gamma_1^*) \cup \dots \cup (-\Gamma_n^*)$$

$$0 \overset{\text{Lem. 7}}{=} 2\pi I_C = \oint_C d\varphi$$

$$= \Big(\int_\Gamma + \underset{-\int_{\Gamma_i}}{\underset{\shortparallel}{\sum_i \int_{-\Gamma_i}}} + \underbrace{\sum_i \int_{\Gamma_i^*} + \sum_i \int_{-\Gamma_i^*}}_{=0} \Big) d\varphi$$

$$= \oint_\Gamma d\varphi - \sum_i \int_{\Gamma_i} d\varphi = 2\pi I_\Gamma - 2\pi \sum_i I_i \qquad \square$$

Eks. 2: Indeks av vilk. orden $n \in \mathbb{N}$

$$\dot{z} = f(z) = z^n \,, \quad n \in \mathbb{N}$$

$$\Updownarrow \; z = x + iy$$

$$(*) \quad \begin{cases} \dot{x} = \operatorname{Re} f(z) = f_1(x,y) \\ \dot{y} = \operatorname{Im} f(z) = f_2(x,y) \end{cases}$$

Begge likn.: $f = 0$ kun i 0

$$I_0 = \frac{1}{2\pi} \oint_\Gamma d\varphi = \frac{1}{2\pi} \oint_\Gamma d(\arg f(z))$$

siden $\varphi = \arctan \frac{f_2}{f_1} = \arg f(z)$.

La $\Gamma_r = \{ re^{is} : s \in [0, 2\pi) \}$ (sirkel).

$$f(z(s)) = r^n e^{ins}$$

$$\Rightarrow \arg f(z(s)) = \arg e^{ins} = ns + 2k\pi$$

$$\Rightarrow \oint_{\Gamma_r} d(\arg f(z)) = \int_0^{2\pi} \frac{d}{ds} \arg f(z(s))\, ds = 2\pi n$$

Dvs. $(0,0)$ likev. pkt. for $(*)$ m. $I_0 = n$.

Obs. 4: Indeks av vilk. orden $-n$, $n \in \mathbb{N}$.

$$\dot{z} = f(z) = \bar{z}^n \,, \; n \in \mathbb{N} \;\leadsto\; I_0 = -n \quad !$$