Motivation: The pendulum

Pendulum equation:

$$\ddot{x} + \omega^2 \sin x = 0. \tag{1}$$

No solution in terms of elementary functions.

Phase plane analysis:

1. Equivalent system ($y = \dot{x}$):

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\omega^2 \sin x. \end{cases} \tag{2}$$

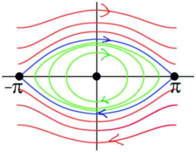
2. Implicit solution = phase trajectory (here C is energy):

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -\frac{\omega^2 \sin x}{y} \quad \Longrightarrow \quad \frac{1}{2}y^2 - \omega^2 \cos x = C = \text{const.}$$ i.e. a curve for each C.

3. Equilibrium points (x_e, y_e) = constant solutions of (2):

$$\Leftrightarrow \ (y_e, -\omega^2 \cos x_e) = (0,0) \Leftrightarrow \ (x_e, y_e) = (n\pi, 0), \ n \in \mathbb{Z}.$$

4. Phase diagram: All phase trajectories (periodic in x, symmetric about $y = 0$)



5. Interpretation: Red = rotations, green = swinging back and forth.

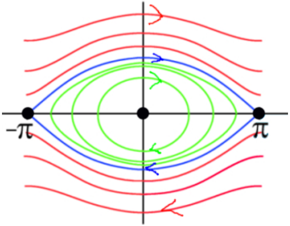
Stabililty:

A solution is stable if all solutions starting suffiently near remain near the solution for all later times.

Stability for the pendulum:

Solutions follow the directions of the arrows, so the phase portrait indicate that:

1. The equilibrium points $((2m + 1)\pi, 0)$ are unstable.
2. The separatrices (blue curve) are unstable.
3. Other trajectories are stable: Equi. pt's $(2m\pi, 0)$, cosine/ellipse like trajectories.



Due to small disturbances, physical systems tend over time to be (near) their stable equilibrium solutions (not neccesarily equilibrium points).

Linear 2×2 systems of ODEs

Initial value problem:

$$(1) \qquad \begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 \\ \dfrac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2, \end{cases} \qquad \text{or} \quad \frac{d\vec{x}}{dt} = A\vec{x} \quad \text{or} \quad \frac{dx}{dt} = Ax,$$

$$(2) \qquad \vec{x}(t_0) = \vec{x}_0.$$

where $t_0 \in \mathbb{R}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$ (constant), and $\vec{x}(t) = x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

Autonomous equation (does not depend on t) in normal form (like $\dot{x} = f(x,t)$).

Results:

1. Theorem 1: There is only one solution of (1) and (2) for all $t \in \mathbb{R}$.
2. Theorem 2: x_1 and x_2 solve (1) $\Rightarrow$ $c_1\vec{x}_1 + c_2\vec{x}_2$ solve (1) for all $c_1, c_2 \in \mathbb{R}$.
3. Theorem 3: x_1 and x_2 solve (1) and are linearly independent
   $\Rightarrow$ any solution of (1) can be written as $c_1\vec{x}_1 + c_2\vec{x}_2$ for some $c_1, c_2 \in \mathbb{R}$.

   Lemma: Solutions x_1 and x_2 of (1) are linearly independent on an interval I if and only if $x_1(t_0)$ and $x_2(t_0)$ are linearly independent for some $t_0 \in I$.

In Tm. 3 x_1, x_2 is a basis, and $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$ a general solution of (1).

Solving equation (1)

(1) $$\frac{d\vec{x}}{dt} = A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Idea: Find a basis of two independent solutions.

Test solution: $\vec{x} = \vec{v}e^{\lambda t}, \quad \vec{v} \neq \vec{0}.$

1. Solves (2) iff $(A - \lambda I)\vec{v} = \vec{0}$ ... Eigenvalue problem.
2. Non-zero solutions $\vec{v}$ iff characteristic equation holds

$$\det(A - \lambda I) = \lambda^2 - p\lambda + q = 0, \quad \text{where} \quad \begin{cases} p = \operatorname{tr} A = a_{11} + a_{22}, \\ q = \det A = a_{11}a_{22} - a_{12}a_{21}. \end{cases}$$

3. Hence always solutions $(\lambda_1, \vec{v}_1)$, $(\lambda_2, \vec{v}_2)$, but complex solutions possible.

Solving linear 2x2 systems

(1) $\dot{x} = Ax$ or $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Test solution: $x = ve^{\lambda t}$, $v \neq 0$.

1. Solves (1) iff $(A - \lambda I)v = 0$ ... Eigenvalue problem ... solutions iff:

   $\det(A - \lambda I) = \lambda^2 - p\lambda + q = 0, \quad q = a_{11} + a_{22}, \quad q = a_{11}a_{22} - a_{12}a_{21}.$

2. Hence always solutions (λ_1, v_1), (λ_2, v_2). All possible cases:
   1. $\lambda_1 \neq \lambda_2$ real: $v_1, v_2 \in \mathbb{R}^2$ are linearly independent.
   2. $\lambda_1 = \lambda_2$ real: $v_1, v_2 \in \mathbb{R}^2$ may or may not be lin. independent.
   3. $\lambda_1 = \bar{\lambda}_2$ complex: $v_1 = \bar{v}_2 \in \mathbb{C}^2$ are linearly independent.

General solution: $x = C_1x_1 + C_2x_2, \quad C_1, C_2 \in \mathbb{R},$ where

1. λ_1, λ_2 real, v_1, v_2 lin. independent: $\boxed{x_1(t) = v_1e^{\lambda_1 t}}$ $\boxed{x_2(t) = v_2e^{\lambda_2 t}}$
2. $\lambda_1 = \lambda_2$ real, only one lin. indep. v: $\boxed{x_1(t) = ve^{\lambda_1 t}}$ and

   $\boxed{x_2 = (vt + u)e^{\lambda_1 t}}$ where $(A - \lambda_1 I)u = v.$
3. $\lambda_1 = \bar{\lambda}_2 = \alpha - i\beta$ complex: $\boxed{x_1(t) = \mathrm{Re}(v_1e^{\lambda_1})}$ $\boxed{x_2(t) = \mathrm{Im}(v_1e^{\lambda_1})}$

Linear 2x2 systems

(1) $$\dot{x} = Ax \quad \text{or} \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

1. Thm: All solutions of (1) can be written in the form

$$x(t) = C_1 x_1(t) + C_2 x_2(t),$$

where x_1 and x_2 are defined as above in each case.

2. Thm: All solutions of (1) are linear combinaitions of products of trigonometric, exponential, and plynomial functions.

3. Thm: The initial value problem (1) and

(2) $$x(0) = x_0,$$

has a solution for all $x_0 \in \mathbb{R}^2$ and all $t \in \mathbb{R}$.

Fundamental matrix and Stability

$$\dot{x}(t) = Ax(t), \tag{1}$$
$$x(t_0) = x_0. \tag{2}$$

where $A \in \mathbb{R}^{2\times 2}$ (constant) and $x : \mathbb{R} \to \mathbb{R}^2$.

Concepts:

1. Fundamental matrix: $\Phi(t) = [x_1(t), x_2(t)]$ where x_1, x_2 basis for (1)
2. Flow ϕ: $x(t) = \phi(t; x_0, t_0)$ solve (1) and (2).
3. Lemma: $\phi(t; x_0, t_0) = \Phi(t)\Phi(t_0)^{-1}x_0$.

Stability: Let $x(t) = \phi(t; x_0, t_0)$.

1. $x(t), t > t_0$, stable if for all $\varepsilon > 0$ there is $\delta > 0$ such that
$$|x_1 - x_0| < \delta \quad \Rightarrow \quad |\phi(t; x_1, t_0) - x(t)| < \varepsilon \quad \text{for all } t > t_0.$$
2. $x(t), t > t_0$, asymptotically stable if stable and there is $\eta > 0$ s.t.
$$|x_1 - x_0| < \eta \quad \Rightarrow \quad |\phi(t; x_1, t_0) - x(t)| \to 0 \text{ as } t \to \infty.$$

Obs: These definitions also hold for general non-linear $n \times n$ systems!

Stability of solutions of linear systems

$$\dot{x}(t) = Ax(t), \tag{1}$$
$$x(0) = x_0. \tag{2}$$

where $A \in \mathbb{R}^{2\times 2}$ (constant) and $x : \mathbb{R} \to \mathbb{R}^2$.

Theorem 1: Let Φ be any fundamental matrix of (1).

1. $\|\Phi(t)\| \leq M < \infty$ for all $t \geq t_0 \quad \Rightarrow \quad$ all solutions of (1) are stable.
2. $\lim_{t\to\infty} \|\Phi(t)\| = 0 \quad \Rightarrow \quad$ all solutions of (1) are asymptot. stable.
3. $\lim_{t\to\infty} \|\Phi(t)\| = \infty \quad \Rightarrow \quad$ all solutions of (1) are unstable.

Theorem 2: Let λ_1, λ_2 be the eigenvalues of A.

1. all solutions of (1) are stable $\quad \Rightarrow \quad \max_i \mathrm{Re}(\lambda_i) \leq 0$.
2. $\max_i \mathrm{Re}(\lambda_i) \leq 0$ and $\lambda_1 \neq \lambda_2 \quad \Rightarrow \quad$ all solutions of (1) are stable.
3. $\max_i \mathrm{Re}(\lambda_i) < 0 \quad \Rightarrow \quad$ all solutions of (1) are asymptot. stable.
4. $\max_i \mathrm{Re}(\lambda_i) > 0 \quad \Rightarrow \quad$ all solutions of (1) are unstable.

Autonomous linear 2x2 systems

(1) $$\dot{x} = Ax; \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Equilibrium points:

- Constant solutions x_e of (1), i.e. solutions of $Ax_e = 0$.
- $\lambda_1, \lambda_2 \in \mathbb{R}$: Node if $\lambda_1\lambda_2 > 0$, and saddle if $\lambda_1\lambda_2 < 0$.
- $\lambda_1 = \bar{\lambda}_2 \in \mathbb{C}$: Spiral if $\operatorname{Re}\lambda_1 \neq 0$, and center if $\operatorname{Re}\lambda_1 = 0$.
- $\lambda_1 = \lambda_2$ or either $\lambda_1 = 0$ or $\lambda_2 = 0$: Degenerate cases.

Phase diagrams/portraits:

1. Phase trajectory through x_0: $\{x(t) : t \in \mathbb{R}, x \text{ solves } (1),\ x(0) = x_0\}$

   Tangent at x: Ax [$= \dot{x}$].

   Direction: Direction of x as time increases = tangent direction
   - Direction in one pt. $\rightarrow$ all directions by continuity of directions

   Equation: $\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{a_{21}x_1 + a_{22}x_2}{a_{11}x_1 + a_{12}x_2}$

   Equilibrium point: Trajectory = one point

2. Phase plane/diagram: x_1x_2-plane/sketch of "all" phase trajectories.

Remark: One trajectory through every $x_0 \in \mathbb{R}^2$; trajectories do not cross!

Linear non-autonoumous nxn systems

$$(1) \qquad \begin{cases} \dot{x} = A(t)x + b(t), \\ x(t_0) = x_0, \end{cases} \qquad A : \mathbb{R} \to \mathbb{R}^{n\times n};\ b, x : \mathbb{R} \to \mathbb{R}^n.$$

We always assume: $A(t)$ and $b(t)$ continuous for all $t \in \mathbb{R}$.

1. Non-homogeneous, non-autonomous equation in normal form.
2. There exists a unique solution of (1) for all $x_0 \in \mathbb{R}^n$ and $t \in \mathbb{R}$.
3. Flow $\phi(t; x_0, t_0)$: The solution of eq'n $(1)_1$ and initial data $(1)_2$.

Homogeneous equation $b \equiv 0$:

$$(2) \qquad \dot{x} = A(t)x$$

1. Basis: n linearly independent solutions $x_1, \ldots, x_n$ of $\dot{x} = Ax$
2. Fundamental matrix Φ: Any for all $t \in \mathbb{R}$ invertible solution of
$$\dot{\Phi} = A\Phi, \qquad A, \Phi : \mathbb{R} \to \mathbb{R}^{n\times n}.$$
3. There always exists a fundamental matrix (and a basis) for (1).
4. **OBS:** $\Phi = [x_1, \ldots, x_n]$ and $b \equiv 0 \Rightarrow \boxed{\phi(t; x_0, t_0) = \Phi(t)\Phi(t_0)^{-1}x_0}$

Linear $n \times n$ systems

Non-autonomous equation $\dot{x} = A(t)x + b(t); \quad A : \mathbb{R} \to \mathbb{R}^{n \times n}$

Φ any fundamental matrix

$$\Rightarrow \boxed{x(t) = \Phi(t)\Phi^{-1}(t_0)x(t_0) + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)b(s)ds}$$

Autonomous equation $\dot{x} = Ax; \quad A \in \mathbb{R}^{n \times n}$ constant

1. $\boxed{x = re^{\lambda t}}$ solve $\dot{x} = Ax$ iff $\boxed{(A - \lambda I)r = 0}$
2. Basis when n linearly independent eigen vectors $r_1, \ldots, r_n$:
$$x_1(t) = r_1 e^{\lambda_1 t}, \ \ldots \ , x_n(t) = r_n e^{\lambda_n t}.$$
3. Basis in general:
$$x_1(t) = p_1(t)e^{\lambda_1 t}, \ \ldots \ , x_n(t) = p_n(t)e^{\lambda_n t},$$
where $p_1, \ldots, p_n : \mathbb{R} \to \mathbb{C}^n$ are polynomials of order $\leq n$.
4. Real basis: $\mathrm{Re}\, x_i$, $\mathrm{Im}\, x_i$, $i = 1, \ldots, n$ (give n lin. indep. solutions).

Jordan form and exponential

Jordan form: $A \in \mathbb{C}^{n\times n}$

1. Eigenvalues/-vectors λ_i/r_i.
2. Complex diagonalization: When n linearly independent r_i's,
$$A = P\Lambda P^{-1} \quad \text{where} \quad \Lambda = \mathrm{diag}(\lambda_i) \in \mathbb{C}^{n\times n}, \quad P = [r_1 \dots r_n].$$
3. Jordan form: For any real matrix A! There is a P s.t.
$$A = P^{-1}JP \quad \text{where} \quad J = \mathrm{diag}(B_1, \dots, B_m) \in \mathbb{R}^{n\times n},$$

$$B_i = \lambda_i, \begin{pmatrix} \mathrm{Re}\,\lambda_i & \mathrm{Im}\,\lambda_i \\ -\mathrm{Im}\,\lambda_i & \mathrm{Re}\,\lambda_i \end{pmatrix} =: D_i, \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ 0 & & & \lambda_i \end{bmatrix}, \text{ or } \begin{bmatrix} D_i & I & & 0 \\ & \ddots & \ddots & \\ & & D_i & I \\ 0 & & & D_i \end{bmatrix}.$$

Matrix exponential: $A \in \mathbb{C}^{n\times n}$

1. $e^A := \lim_{n\to\infty} \left(I + A + \frac{1}{2}A^2 + \cdots + \frac{1}{n!}A^n \right)$.
2. e^A is well-defined and $\|e^A\| \leq e^{\|A\|}$.
3. $\Phi(t) = e^{tA}$ solve the following initial value problem
$$\dot{\Phi} = A\Phi \qquad \text{and} \qquad \Phi(0) = I.$$

Matrix exponential and stability

(1) $\dot{x} = A(t)x + b(t); \quad A : \mathbb{R} \to \mathbb{R}^{n\times n}; \; b, x : \mathbb{R} \to \mathbb{R}^n.$

Matrix exponential:

1. $A = PJP^{-1} \quad \Rightarrow \quad e^A = Pe^J P^{-1}$
2. e^{tA} fundamental matrix; the solution of (1) when $A = \mathrm{konst}$ is
$$x(t) = e^{(t-t_0)A}x(t_0) + \int_{t_0}^{t} e^{(t-s)A}b(s)ds.$$

Stability:

1. Theorem: Φ fundamental matrix of A, $\dot{\Phi} = A(t)\Phi$.
   (a) Φ bounded, $t \geq t_0 \Rightarrow$ all solutions of (1) stable
   (b) $\Phi \to 0$ as $t \to \infty \Rightarrow$ all solutions of (1) asympt. stable
   (c) Φ unbounded, $t \geq t_0 \Rightarrow$ all solutions of (1) unstable
2. Theorem: $A = \mathrm{konst}$, eigenvalues λ_i.
   (a) $\max_i \mathrm{Re}\,\lambda_i \leq 0$ and $\lambda_i \neq \lambda_j \Rightarrow$ all solutions of (1) stable
   (b) $\max_i \mathrm{Re}\,\lambda_i < 0 \Rightarrow$ all solutions of (1) asympt. stable
   (c) $\max_i \mathrm{Re}\,\lambda_i > 0 \Rightarrow$ all solutions of (1) unstable

Stability when A is non-constant

(2) $$\dot{x} = A(t)x + b(t); \quad A : \mathbb{R} \to \mathbb{R}^{n \times n}; \; b, x : \mathbb{R} \to \mathbb{R}^n.$$

1. Assume: $A(t) = B + C(t)$ where B constant, $\int_{t_0}^{\infty} \|C(s)\| ds < \infty$.
2. Theorem: Φ fundamental matrix for B, $\dot{\Phi} = B\Phi$.
   (a) Φ bounded, $t \geq t_0 \Rightarrow$ all solutions of (1) stable
   (b) $\Phi \to 0$ as $t \to \infty \Rightarrow$ all solutions of (1) asympt. stable
3. Theorem: B has eigenvalues λ_i.
   (a) $\max_i \operatorname{Re} \lambda_i \leq 0$ and $\lambda_i \neq \lambda_j \Rightarrow$ all solutions of (1) stable
   (b) $\max_i \operatorname{Re} \lambda_i < 0 \Rightarrow$ all solutions of (1) asympt. stable
4. Grönwall's Lemma
   Let $u, v : \mathbb{R} \to \mathbb{R}$ be continuous, ≥ 0, $K > 0$, and

$$u(t) \leq K + \int_{t_0}^{t} v(s)u(s)\, ds \qquad \text{for} \qquad t > t_0,$$

then

$$u(t) \leq K e^{\int_{t_0}^{t} v(s)\, ds} \qquad \text{for} \qquad t > t_0.$$

Stability when A is non-constant

(1) $$\dot{x} = A(t)x + b(t); \quad A : \mathbb{R} \to \mathbb{R}^{n\times n};\ b, x : \mathbb{R} \to \mathbb{R}^n.$$

1. Assume: $A(t) = B + C(t)$ where B constant, $\int_{t_0}^{\infty} \|C(s)\| ds < \infty$.

2. Theorem: Φ fundamental matrix for B, $\dot{\Phi} = B\Phi$.

   (a) Φ bounded, $t \geq t_0 \Rightarrow$ all solutions of (1) stable

   (b) $\Phi \to 0$ as $t \to \infty \Rightarrow$ all solutions of (1) asympt. stable

3. Theorem: B has eigenvalues λ_i.

   (a) $\max_i \operatorname{Re} \lambda_i \leq 0$ and $\lambda_i \neq \lambda_j \Rightarrow$ all solutions of (1) stable

   (b) $\max_i \operatorname{Re} \lambda_i < 0 \Rightarrow$ all solutions of (1) asympt. stable

Lipschitz condition

Definitions: Let $f = (f_1, \dots, f_n) : R \to \mathbb{R}^n, \quad R \subset \mathbb{R}^{n+1}$.

1. $f(x)$ is Lipschitz in a set $\Omega \subset \mathbb{R}^n$ if there is $L > 0$ such that
$$|f(x) - f(y)| \le L|x - y| \quad \text{for all} \quad x, y \in \Omega.$$
2. $f(x,t)$ is x-Lipschitz in a set $R \subset \mathbb{R}^{n+1}$ if there is $L > 0$ such that
$$|f(x,t) - f(y,t)| \le L|x - y| \quad \text{for all} \quad (x,t), (y,t) \in R.$$
3. $f(x)$ is locally Lipschitz on Ω if they are Lipschitz on each closed bounded subset $K \subset \Omega$ (or – on each closed bounded ball $\overline{B} \subset \Omega$).
4. Locally x-Lipschitz defined similarly.
5. Lipschitz constant: The smallest L in the above inequalities.
6. $[\sqrt{x_1^2 + x_3^2}, 5, \frac{x_1^2}{1+x_1^2+x_3^2}]$ Lip. in $\mathbb{R}^3$; $\quad x^2$ only loc. Lip./$\sqrt{x}$ not in $\mathbb{R}$

Lipschitz condition

1. $f(x,t)$ is x-Lipschitz in a set $R \subset \mathbb{R}^{n+1}$ if there is $L > 0$ such that
$$|f(x,t) - f(y,t)| \le L|x-y| \quad \text{for all} \quad (x,t),(y,t) \in R.$$
2. $f(x,t)$ is locally x-Lipschitz in R if it is x-Lipschitz on each closed bounded subset $K \subset \Omega$ (or closed ball $\overline{B} \subset \Omega$).
3. Jacobi matrix: $Df(x,t) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$
4. f is C^1 in closed, bnd, convex $R \Rightarrow f$ x-Lip. in R, $L \le \max_R \|Df\|$.

Uniqueness and continuous dependence

$$\dot{x}(t) = f(x(t), t), \tag{1}$$

$$x(t_0) = x_0. \tag{2}$$

Uniqueness:

If f is continuous and locally x-Lipschitz in a domain R, then there are not more than one solution of (1) and (2) in R.

Continuous dependence on initial data:

Let $R \subset \mathbb{R}^{N+1}$ be a domain. If

1. f is continuous and x-Lipschitz in R,
2. $(x(t), t)$, $(y(t), t) \subset R$ for $t \in [t_0, b)$,
3. x and y solve

$$\begin{cases} \dot{x} = f(x, t) & \text{in } (t_0, b) \\ x(t_0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{y} = f(y, t) & \text{in } (t_0, b) \\ y(t_0) = y_0, \end{cases}$$

then $|x(t) - y(t)| \leq e^{L(t-t_0)}|x_0 - y_0|$ in $t \in [t_0, b)$.

Remark: $x(t) = \phi(t; x_0, t_0)$ depends (locally Lipschitz) continuous on x_0.

Continuous dependence on the data

Theorem:

Let $R \subset \mathbb{R}^{N+1}$ be a domain. If

1. f is continuous and x-Lipschitz in R,
2. g is continuous and there is $\varepsilon > 0$ s.t. $|f(x,t) - g(x,t)| \leq \varepsilon$ in R,
3. $(x(t),t)$, $(y(t),t) \subset R$ for $t \in [t_0, b)$,
4. x and y solve

$$\begin{cases} \dot{x} = f(x,t) & \text{in } (t_0,b) \\ x(t_0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{y} = g(y,t) & \text{in } (t_0,b) \\ y(t_0) = y_0, \end{cases}$$

then

$$|x(t) - y(t)| \leq e^{2L(t-t_0)}|x_0 - y_0| + \frac{\varepsilon}{L}\sqrt{e^{2L(t-t_0)} - 1} \quad \text{in } [t_0, b),$$

where $L \leq \max_R \|Df(x,t)\|$.

Remark: The solution $x(t) = \phi(t; x_0, t_0, f)$ is continuous in x_0, f (, t_0!).

Integral equation and Piccard interations

$$(1) \qquad \dot{x}(t) = f(x(t), t),$$
$$(2) \qquad x(t_0) = x_0.$$

Lemma: If f is continous, then $x(t)$ solves (1) and (2) if and only if $x(t)$ is a continuous solution of (3):

$$(3) \qquad x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds.$$

Piccard iterations:

$$\begin{aligned} x^0(t) &= x_0 \\ x^1(t) &= x_0 + \int_{t_0}^{t} f(x^0(s), s)ds \\ &\vdots \\ x^k(t) &= x_0 + \int_{t_0}^{t} f(x^{k-1}(s), s)ds \end{aligned}$$

Idea: x_k is an approximation of solution x of (3) and $x^k \to x$ uniformly.
[$\{x_k\}$ Cauchy sequence in $C[t_0 - T, t_0 + T]$]

Existence of solutions

$$(1) \qquad \dot{x}(t) = f(x(t), t),$$
$$(2) \qquad x(t_0) = x_0.$$

Theorem 1: ("Global" existence)

If f is continuous and x-Lipschitz for $x \in \mathbb{R}^n$ and $|t - t_0| \leq T$, then there exists a solution of (1) and (2) for $|t - t_0| \leq T$.

Theorem 2: (Local existence 1 – Piccard-Lindelöf)

If f is continuous + x-Lipschitz for $|x - x_0| \leq K$, $|t - t_0| \leq T$, then there exists a sol'n of (1) + (2) for $|t - t_0| \leq \min(T, \frac{K}{M})$, $M = \max\limits_{\substack{|t - t_0| \leq T \\ |x - x_0| \leq K}} |f(x,t)|$

Theorem 3: (Local existence 2 – Peano)

If f is continuous for $|x - x_0| \leq K$ and $|t - t_0| \leq T$, then there exists a solution of (1) and (2) for $|t - t_0| \leq \min(T, \frac{K}{M})$ where M as in Thm. 2.

Remark: In Thm's 1 and 2, the solution is unique – but not in Thm 3!

Existence interval and C^1 dependence on x_0.

$$\dot{x}(t) = f(x(t), t), \tag{1}$$

$$x(t_0) = x_0. \tag{2}$$

Theorem 1: f continuous and x-Lipschitz on domain $R \ni (x_0, t_0)$.

(a) There exists a unique solution of (1) and (2) on a maximal existence interval $(t_0 - a, t_0 + b)$ for some $a, b > 0$.

(b) If $b < \infty$, then either $(x(t), t) \to \partial R$ or $|x(t)| \to \infty$ as $t \to b^-$.

Remark:

1. Either $b = \infty$, or x blows up in finite time, or $(x(t), t)$ leaves in finite time the region R of well-posedness.
2. b = existence/life time of the solution of (1) and (2).

Theorem 2: $f \in C^1$ and $\phi(t; x_0)$ (unique) solution of (1) and (2).

Then ϕ is C^1 in x_0 (and t), and $w = \frac{\partial \phi}{\partial x_j}$ is the unique solution of

$$w_t = Df(\phi(t; x_0), t)w, \quad w(0) = e_j.$$

Phasediagram for autonomous systems

(1) $\dot{x} = f(x), \quad f \in R^n$ independent of t

(2) $x(t_0) = x_0.$

1. Phase trajectory through x_0: $\{x(t) : t \in \mathbb{R}, x \text{ solves } (1), x(0) = x_0\}$
   Tangent (direction) at x: $f(x)$ $[= \dot{x}]$.
   - f continuous $\Rightarrow$ continuity of directions

   Equation: $\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{f_2(x)}{f_1(x)}, \ldots, \frac{dx_n}{dx_1} = \frac{\dot{x}_n}{\dot{x}_1} = \frac{f_n(x)}{f_1(x)}$

   Equilibrium point x_e: $f(x_e) = 0$, and trajectory = one point
2. Phase diagram: Sketch of "all" phase trajectories.
3. Well-posedness for (1) and (2) for all x_0, t
   $\Rightarrow$ trajectories exist + do not cross + pass through every $x_0 \in \mathbb{R}^n$.
4. Separatrix: Trajectory separating regions w. different sol'n behaviour
5. Only in non-linear systems: Multiple isolated equilibrium points, limit cycles, separatix cycles, and chaos (in $\mathbb{R}^n$, $n \geq 3$).

Linearization

(1) $$\dot{x} = f(x).$$

Linearized about equi. pt. x_0: $f(x) = f(x_0) + Df(x_0)(x - x_0) + \dots$

(2) $$\dot{y} = Df(x_0)y \quad (y \approx x - x_0).$$

1. Near x_0, $f(x_0) \neq 0$: Phase diagram $\approx$ straigth lines (if $f \in C^1$).
2. Near x_0, $f(x_0) = 0$ and hyperbolic: $\operatorname{Re} \lambda_i \neq 0$ for eig.val's of $Df(x_0)$.
   1. Phasediagram (1) near $x = x_0$ $\approx$ phasediagram (2) near $y = 0$.
   2. Same type of equilibrium point in $x = x_0$ and $y = 0$.
   3. Justification: Hartman-Grobman + Hartman thm's (needs $f \in C^2$).
3. Near x_0, $f(x_0) = 0$ and non-hyperbolic:

   No information from linearization – need higher order theory.

Linearization and Hartman-Grobman

(1) $$\dot{x} = f(x).$$

Linearized about equi. pt. x_0: $f(x) = f(x_0) + Df(x_0)(x - x_0) + \dots$

(2) $$\dot{y} = Df(x_0)y \quad (x - x_0 = y).$$

1. Near x_0, $f(x_0) \neq 0$: Phase diagram $\approx$ straigth lines (if $f \in C^1$).
2. Near x_0, $f(x_0) = 0$ and hyperbolic: $\operatorname{Re} \lambda_i \neq 0$ for eig.val's of $Df(x_0)$
   1. Phase diagram (1) looks like phase diagram (2) near $y = 0$
   2. (1) and (2) has same types of equilibrium points in x_0 and $y = 0$ (including asymptotic lines etc.)
   3. Justification: Hartman-Grobman + Hartman thm's (need $f \in C^2$!).
3. Near x_0, $f(x_0) = 0$ and non-hyperbolic:

   No information from linearization – need higher order theory.

Reminder: Stability

$$\dot{x}(t) = f(x(t), t), \tag{1}$$

$$x(t_0) = x_0. \tag{2}$$

Flow ϕ: $x(t) = \phi(t; x_0, t_0)$ solution of (1) and (2).

Stability: Let $x(t) = \phi(t; x_0, t_0)$.

1. $x(t)$ stable for $t \geq t_0$ if for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$|x_1 - x_0| < \delta \quad \Rightarrow \quad |\phi(t; x_1, t_0) - \phi(t; x_0, t_0)| < \varepsilon \quad \text{for all } t > t_0.$$

2. $x(t)$ asymptotic stable for $t \geq t_0$ if stable and there is $\eta > 0$ s.t.

$$|x_1 - x_0| < \eta \quad \Rightarrow \quad |\phi(t; x_1, t_0) - \phi(t; x_0, t_0)| \to 0 \text{ as } t \to \infty.$$

Stability of equilibrium points

Nearly linear systems:

(1) $\dot{x} = Ax + h(x,t); \qquad A \in \mathbb{R}^{n\times n}, \quad x : \mathbb{R} \to \mathbb{R}^n, \quad h : \mathbb{R}^{n+1} \to \mathbb{R}^n.$

Theorem 1:

Assume λ_i eigenvalues of A, $h \in C^1$, and

$$|h(x,t)| = o(|x|) \quad \text{as} \quad x \to 0 \quad \text{uniformly in } t.$$

$\max_i \operatorname{Re} \lambda_i < 0 \;\Rightarrow\; 0$ is an asympt. stable equilibrium pt. of (1).

Autonomous systems:

(2) $\dot{x} = f(x); \qquad x, f : \mathbb{R} \to \mathbb{R}^n.$

Theorem 2: (Linearization)

Assume $f \in C^1$, $f(x_0) = 0$, and λ_i eigenvalues of $Df(x_0)$.

(a) $\max_i \operatorname{Re} \lambda_i < 0 \;\Rightarrow\; x_0$ is an asympt. stable equilibrium pt. of (2)

(b) $\max_i \operatorname{Re} \lambda_i > 0 \;\Rightarrow\; x_0$ is an unstable equilibrium pt. of (2)

(c) $\max_i \operatorname{Re} \lambda_i = 0 \;\Rightarrow$ NO CONCLUSION!!

OBS: Stability based on Df = stability by linearization, $\dot{y} = Df(x_0)y$.

Liapunov's direct method

- More general method than linearization.
- Need to construct (find) a Liapunov function.

$$\dot{x} = f(x); \qquad x, f : \mathbb{R} \to \mathbb{R}^n. \tag{2}$$

Definition: $V(x)$ is a strong Liapunov function for (2) in domain $B \ni 0$ if

1. $V \in C^1(B)$,
2. $V(0) = 0$ and $V(x) > 0$ for $x \in B \setminus \{0\}$,
3. $\dot{V}(0) = 0$ and $\dot{V}(x) < 0$ for $x \in B \setminus \{0\}$, where

$$\dot{V}(x) = \frac{d}{dt} V(x(t)) = \nabla V(x) \cdot \dot{x} = \nabla V(x) \cdot f(x).$$

Theorem 2:

Assume $f(0) = 0$ and f Lipschitz in some domain $B \ni 0$.

If $V(x)$ is a strong Liapunov function for (2) in B, then $x = 0$ is an asymptotically stable equilibrium point for (2).

Liapunov's direct method

(1) $\dot{x} = f(x); \qquad x : \mathbb{R} \to \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n.$

Weak (strong) Liapunov function $V(x)$ for (1) in domain $B \ni 0$:

1. $V \in C^1(B)$,
2. $V(0) = 0$ and $V(x) > 0$ for $x \in B \setminus \{0\}$,
3. $\dot{V}(0) = 0$ and $\dot{V}(x) \leq 0$ $(V(x) < 0)$ for $x \in B \setminus \{0\}$, where
4. $\dot{V}(x) = \frac{d}{dt} V(x(t)) = \nabla V(x) \cdot \dot{x} = \nabla V(x) \cdot f(x)$.

Candidates:

$V(x) = \sum_i c_i x_i^2$, or $V(x) = x^T A x$ for $A = A^T$ positive definite, or $V =$ energy/Hamiltonian etc.

Theorem: Assume $f(0) = 0$ and f Lipschitz in some domain $B \ni 0$.

(a) If $V(x)$ is a weak Liapunov function for (1) in B, then $x = 0$ is an stable equilibrium point for (1).

(b) If $V(x)$ is a strong Liapunov function for (1) in B, then $x = 0$ is an asymptotically stable equilibrium point for (1).

Liapunov methods for autonomous systems

(1) $$\dot{x} = f(x); \qquad f, x \in \mathbb{R}^n.$$

Definitions:

1. $\dot{V}(x) = \frac{d}{dt}V(x(t)) = \nabla V(x) \cdot f(x)$
2. $V(x)$ weak (strong) Liapunov function of (1) in domain $B \ni 0$ if
   1. $V \in C^1(B)$,
   2. $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$
   3. $\dot{V}(0) = 0$ and $\dot{V}(x) \leq 0$ ($\dot{V}(x) < 0$) for $x \neq 0$

Theorem: Assume $f(0) = 0$ and f Lipschitz in a domain $B \ni 0$.

1. $V(x)$ weak Lipunov function for (1) $\Rightarrow x(t) \equiv 0$ is stable
2. $V(x)$ strong Lipunov function for (1) $\Rightarrow x(t) \equiv 0$ is asympt. stable
3. $U(x)$ satisfy 1–3 below $\Rightarrow x(t) \equiv 0$ is unstable
   1. $U \in C^1(B)$,
   2. $U(0) = 0$; for any $\delta > 0$, there is $x \in B$ s.t. $|x| < \delta$ and $U(x) > 0$,
   3. there is $\eta > 0$ such that $\dot{U}(x) > 0$ for all x s.t. $|x| < \eta$ and $U(x) > 0$.

Examples: $V = \frac{1}{2}(x_1^2 + x_2^2)$, = energy/Hamiltonian; $U = x_1 x_2$, $= x_1^2 - x_2^2$

Invariant domains and domains of attraction

$$(1) \qquad \dot{x} = f(x); \qquad f, x \in \mathbb{R}^n.$$

1. Flow: $\phi(t; x_0)$ solution of (1) and $x(0) = x_0$.
2. $\Omega \subset \mathbb{R}$ invariant (positive invariant) under (1)

   if $\phi(t; x_0) \in \Omega$ for all $t \in \mathbb{R}$ $(t \geq 0)$ and all $x_0 \in \Omega$.

   $\Omega_a = \{x \in \mathbb{R}^n : \lim_{t\to\infty} \phi(t; x) = x_0\}$ domain of attraction of x_0.
3. Lemma: $\Omega = \{x : V(x) \leq c\}$ is positive invariant if $f, V \in C^1$ and

   $$\nabla V(x) \neq 0 \quad \text{and} \quad \dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0 \quad \text{on} \quad V(x) = c.$$
4. Lasalle's invariance principle. Assume:
   1. Ω positive invariant, closed and bounded, $0 \in \Omega$.
   2. $f(0) = 0$ and f Lipschitz in Ω,
   3. $V(x)$ is a weak Liapunov function on Ω,
   4. there is no global in time solution of (1) such that

      $$x(t) \in \Omega \setminus \{0\} \quad \text{and} \quad V(x(t)) = \text{constant} \quad \text{for all} \quad t \in \mathbb{R}.$$

   Then $x = 0$ is asymptotically stable and $\Omega \subset \Omega_a$ (attraction).

Hamiltonian 2×2 systems

$$(1) \qquad \dot{x} = f(x); \qquad x = (x_1, x_2), \quad f = (f_1, f_2).$$

Definition: (1) Hamiltonian if there is a Hamilton function $H(x) \in C^2$ s.t.

$$f_1 = \frac{\partial H}{\partial x_2} \quad \text{and} \quad f_2 = -\frac{\partial H}{\partial x_1}.$$

1. (1) Hamiltonian $\quad \Leftrightarrow \quad \nabla \cdot f = 0 \quad$ (f divergence free)
2. H Hamilton function, $x(t)$ solution of (1) $\Rightarrow$ $H(x(t)) =$ constant
3. Equilibrium points of (1) = critical points of H ($\nabla H = 0$)
4. Classification of equilibrium point x_0 via 2nd derivative test for H:

   $q(x_0) > 0$ center, $q(x_0) < 0$ saddle, $q(x_0) = 0$ no conclusion,

   where $q = \det D^2 H = H_{x_1 x_1} H_{x_2 x_2} - H_{x_1 x_2} H_{x_2 x_1} = \lambda_1(D^2 H)\lambda_2(D^2 H)$.

OBS: Similar results hold for $2n \times 2n$ systems: $f_1 = \nabla_{x_2} H$, $f_2 = -\nabla_{x_1} H$

Index theory for 2×2 systems

$$(1) \qquad \dot{x} = f(x); \qquad x, f \in \mathbb{R}^2.$$

1. Polar angle of f: $\phi = \arctan \frac{f_2}{f_1}$
2. Curve Γ: Simple, closed, p.w. C^1, oriented counter cl.wise, $f|_\Gamma \neq 0$
3. Index of Γ: $\boxed{I_\Gamma = \frac{1}{2\pi} \oint_\Gamma d\phi}$
4. $\Gamma = \{y(s) : s_0 \leq s \leq s_T\}$:
$$I_\Gamma = \frac{1}{2\pi} \int_{s_0}^{s_T} \frac{f_1 \frac{d}{ds} f_2 - f_2 \frac{d}{ds} f_1}{f_1^2 + f_2^2} ds; \quad f_i(s) = f_i(y(s)),$$
$$I_\Gamma = \frac{1}{2\pi} \int_{y(s_0)}^{y(s_T)} \nabla\phi \cdot dy = \frac{\phi(y(s_T)) - \phi(y(s_0))}{2\pi} \in \mathbb{Z} \text{ since } f(s_0) = f(s_T)$$
5. Winding number of Γ:
$$W_\Gamma = \frac{1}{2\pi} \oint_\Gamma \frac{X_1 dX_2 - X_2 dX_1}{X_1^2 + X_2^2} = \text{no. of times } \Gamma \text{ winds about } 0 \text{ c. cl.wisely}$$
6. $I_\Gamma = W_{\Gamma_f}$ when $\Gamma_f = \{f(x) : x \in \Gamma\}$ $\qquad [X(s) = f(y(s))]$
7. $\boxed{I_\Gamma = W_{\Gamma_f} = \frac{1}{2}(p - q)}$ where p (q) is number of times $f(x)$ parallel with any given axis and crosses it counter clockwisely (clockwisely) as x traverses Γ counter clockwisely.

Index theory for 2×2 systems

$$(1) \qquad \dot{x} = f(x); \qquad x = (x_1, x_2),\ f = (f_1, f_2).$$

Definition:

1. Index of curve Γ: $I_\Gamma = \frac{1}{2\pi} \oint_\Gamma d\phi$
2. Index of equilibrium point x_e:
   $I_{x_e} = I_\Gamma$ for any Γ encircling x_e but no other equilibrium point of (1).

Remarks:

1. $\phi = \arctan \frac{f_2}{f_1}$ polar angle of f.
2. Γ: Simple, closed, p.w. C^1 curve oriented counter cl.wise, $f|_\Gamma \neq 0$.
3. Saddle: $I = -1$; node, spiral, center, closed phase trajectory: $I = 1$
4. There are equilibrium points with $I = n$ for any $n \in \mathbb{Z}$.

Theorem: Assume f is C^2 and $f = 0$ in Ω only at $x_1, \ldots, x_n$. Then

$$I_\Gamma = I_{x_1} + I_{x_2} + \cdots + I_{x_n}$$

for any curve Γ in Ω encircling $x_1, \ldots, x_n$.

Closed trajectories and Poincare sequences

$$\dot{x} = f(x); \quad x, f \in \mathbb{R}^2. \tag{1}$$

Closed trajectories

1. Closed phase trajectories $\Leftrightarrow$ periodic solutions (autonomous syst.)
2. Limit cycles: Isolated closed phase trajectories.
3. Index test: Closed curve Γ surround equilibrium points x_i, index I_i:
$$\sum I_i \neq 1 \quad \Rightarrow \quad \Gamma \quad \text{is not a phase trajectory.}$$
4. Dulac's test: Ω open, simply connected; $\rho, f = (f_1, f_2) \in C^1(\Omega)$:
$$\nabla \cdot (\rho f) < 0 \text{ in } \Omega \text{ (or } > 0) \quad \Rightarrow \quad \text{no closed phase trajectory in } \Omega.$$
5. $\rho \equiv 1 \rightarrow$ Bendixon's negative criterion

Poincare sequences:

1. Poincare cross section Σ: curve transversal (=non-parallel) to f
2. Poincare map P_Σ of $x_0 \in \Sigma$: Point of first return of flow $\phi(t; x_0)$ to Σ
3. Poincare sequence: $x_0, P_\Sigma(x_0), \dots, P_\Sigma^n(x_0), \dots$ ($P^n = P \circ P \circ \cdots \circ P$)

Poincare Bendixon's theorem

(1) $$\dot{x} = f(x); \qquad x : \mathbb{R} \to \mathbb{R}^2, \quad f : \mathbb{R}^2 \to \mathbb{R}^2.$$

1. $\Gamma_{x_0}^{\pm} = \{\phi(t; x_0) : t \in [0, \pm\infty)\}$ $\quad$ ($\Gamma_{x_0} = \Gamma_{x_0}^{+} \cup \Gamma_{x_0}^{-}$ trajectory; ϕ flow)
2. $\Omega \subset \mathbb{R}^2$ positive invariant under (1): $\Gamma_{x_0}^{+} \subset \Omega$ for all $x_0 \in \Omega$.
3. ω-limit set of Γ_{x_0}: All z s.t. $x(t_n) = \phi(t_n; x_0) \underset{t_n \to \infty}{\to} z$ for some $\{t_n\}_n$.
4. Cycle: A periodic trajectory but no equi. point (limit or center c.).

Theorem (Poincare-Bendixson):
$\Gamma^{+} \subset$ closed bounded K; no equilibrium points in $\omega_\Gamma \;\Rightarrow\; \omega_\Gamma$ is a cycle.

Lemma 1: $\Gamma^{+} \subset K$, K closed, bounded (=compact)
$\Rightarrow \; \omega_\Gamma \subset K$, $\neq \emptyset$, closed, bounded, connected, invariant under (1).

Lemma 2: Assume L transversal line segment ($L \not\parallel f$)

1. Γ crosses L in two different points $\Leftrightarrow$ Γ is not closed.
2. Crossing points of Γ and L are ordered the same way along L as along Γ.

Corollary: $x_0 \in \omega_\Gamma \cap \Gamma \;\Leftrightarrow\; \Gamma$ closed trajectory.

Application of Poincare Bendixon's theorem

(1) $$\dot{x} = f(x); \qquad x : \mathbb{R} \to \mathbb{R}^2, \quad f : \mathbb{R}^2 \to \mathbb{R}^2.$$

Theorem (Poincare-Bendixson):
$\Gamma^+ \subset$ closed bounded K; no equilibrium points in $\omega_\Gamma \;\Rightarrow\; \omega_\Gamma$ is a cycle.

Cycle: A periodic trajectory but not equilibrium point (limit or center c.)
Corresponds to a periodic solution.

Corollary 1: $K \subset \mathbb{R}^2$ closed, bounded, pos. invariant, with no equi. pt's
$\Rightarrow$ at least one cycle in K.

Lemma: $\Omega = \{x : V(x) \leq c\}$ is positive invariant if $f, V \in C^1$ and

$$\nabla V(x) \neq 0 \quad \text{and} \quad \dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0 \quad \text{on} \quad V(x) = c.$$

Corollary 2: There is at least one cycle in $K = \{x : c_1 \leq V(x) \leq c_2\}$ if

1. K is bounded, contains no equilibrium points,
2. $\nabla V \neq 0$ and $\dot{V} = \nabla V \cdot f \geq 0$ on $V(x) = c_1$, and
3. $\nabla V \neq 0$ and $\dot{V} = \nabla V \cdot f \leq 0$ on $V(x) = c_2$.

The Lienard equation

$$(2) \qquad \ddot{x} + f(x,\dot{x})\dot{x} + g(x) = 0 \quad \Leftrightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -f(x_1,x_2)x_2 - g(x_1) \end{cases}$$

Obs:

1. Energy: $E(x_1,x_2) = \frac{1}{2}x_2^2 + G(x_1)$, where $G(x) = \int_0^x g(s)ds$.
2. $\frac{d}{dt}E(x(t)) = \nabla E(x) \cdot \dot{x} = -f(x_1,x_2)x_2^2 \leq 0$ (≥ 0) if $f \geq 0$ ($f \leq 0$)

Theorem (cycles): Equation (2) has at least one cycle if

(a) $f(x_1,x_2)$ continuous, $f < 0$ for $|x| < r$, $f > 0$ for $|x| > R$,

(b) $g(x_1)$ continuous, $g < 0$ for $x_1 < 0$, $g > 0$ for $x_1 > 0$,

(c) $G(x_1) = \int_0^{x_1} g(s)ds \to \infty$ as $|x_1| \to \infty$.

Idea: $x = 0$ only equilibrium point, $E(x) = c_1$ and $E(x) = c_2$ bounds bounded invariant region for c_1 small and c_2 big, use Poincare-Bendixon.

The Lienard equation

(1) $$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0 \quad \Leftrightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -f(x_1, x_2)x_2 - g(x_1) \end{cases}$$

Energy: $E = \frac{1}{2}x_2^2 + \int_0^x g(s)ds$; $\dot{E} = -f(x_1, x_2)x_2^2$.

Theorem (cycle): Equation (1) has at least one cycle if

(a) $f(x_1, x_2)$ continuous, $f < 0$ for $|x| < r$, $f > 0$ for $|x| > R$,

(b) $g(x_1)$ continuous, $g < 0$ for $x_1 < 0$, $g > 0$ for $x_1 > 0$,

(c) $G(x_1) = \int_0^{x_1} g(s)ds \to \infty$ as $|x_1| \to \infty$.

Theorem (centre): Equation (1) has a centre at $x = 0$ if for $|x| < R$:

(a) $f = f(x_1)$ continuous, odd, one sign for $x_1 < 0$,

(b) $g(x_1)$ continuous, odd, $g > 0$ for $x_1 > 0$,

(c) $g(x_1) > \alpha f(x_1) \int_0^{x_1} f(s)ds$ for $\alpha > 1$.

Obs: Trajectories loose energy in $\{x_1 \geq 0\}$ but gain same amount in $\{x_1 < 0\}$.

Theorem (limit cycle): Equation (1) has one and only one cycle if

(a) $F(x) = \int_0^x f(s)ds$, g locally Lipschitz,

(b) g odd, $g > 0$ for $x > 0$,

(c) F odd, $= 0$ only at $x = 0, \pm a$, $F \underset{x\to\infty}{\to} \infty$, monotonically for $x > a$.

Obs: The proof uses the Lienard plane: $\dot{x}_1 = x_2 - F(x_1)$, $\dot{x}_2 = -g(x_1)$