Diff. eq'ns 4.4.2013

A. Hamiltonian syst.

(1) $\quad \dot{x} = f(x)$

where $x = (x_1, x_2)$, $f = (f_1, f_2) \in C^1(\mathbb{R}^2; \mathbb{R}^2)$.

Def. 1:

a) (1) Hamiltonian syst. if exists $H \in C^1(\mathbb{R}^2; \mathbb{R})$

s.t. $\quad f_1 = \frac{\partial H}{\partial x_2} \quad$ and $\quad f_2 = -\frac{\partial H}{\partial x_1}$

b) H = Hamilton func. for (1)

Lem. 1:

(1) Hamiltonian $\iff \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 \quad (\operatorname{div} f = 0)$

[$\Rightarrow$) $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \frac{\partial}{\partial x_1}\frac{\partial H}{\partial x_2} + \frac{\partial}{\partial x_2}\left(-\frac{\partial H}{\partial x_1}\right) = 0$

$\Leftarrow$) $H = \int_0^{x_2} f_1(x_1, s)\, ds - \int_0^{x_1} f_2(s, 0)\, ds$ Ham. func.

when $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0$ (chk.)]

$\left(\frac{\partial H}{\partial x_1} = \int_0^{x_2} \frac{\partial f_1}{\partial x_1}(x_1, s)\, ds - f_2(x_1, x_2) = \int_0^{x_2}\left(-\frac{\partial f_2}{\partial x_2}\right)(x_1, s)\, ds - f_2 = f_2 \right)$

Obs. 1:

i) H Ham. func. $\Rightarrow$ $H + \text{const.}$ Ham. func.

ii) $\int_0^{x_2} f_1(x_1, s)\,ds - \int_0^{x_1} f_2(s, 0)\,ds$ Ham. func. for (1)

iii) (1) Ham. $\Leftrightarrow$ f div. free ($\vec{\nabla}\cdot f = 0$)

Ex. 1:

$$(*)\quad \begin{cases} \dot{x}_1 = x_2(13 - x_1^2 - x_2^2) = f_1 \\ \dot{x}_2 = 12 - x_1(13 - x_1^2 - x_2^2) = f_2 \end{cases}$$

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_2(-2x_1) + (-x_1)(-2x_2) = 0$$

$\Rightarrow$ $(*)$ Ham.

$$\frac{\partial H}{\partial x_2} = f_1 \Rightarrow H = \int_0^{x_2} f_1(x_1, s)\,ds + C(x_1)$$

$$= \frac{13}{2}x_2^2 - \frac{1}{2}x_1^2 x_2^2 - \frac{1}{4}x_2^4 + C(x_1)$$

$$f_2 = -\frac{\partial H}{\partial x_1} = x_1 x_2^2 - C'(x_1)$$

$$\overset{(*)}{\Rightarrow} C'(x_1) = -12 + x_1(13 - x_1^2)$$

$$\Rightarrow C(x_1) = -12x_1 + \frac{13}{2}x_1^2 - \frac{1}{4}x_1^4 + C$$

Take $C = 0$,

$$H = \int_0^{x_2} f_1 + C(x_1)$$

$$= -12x_1 + \frac{13}{2}(x_1^2 + x_2^2) - \frac{1}{4}(x_1^4 + x_2^4) - \frac{1}{2}x_1^2 x_2^2$$

Lem. 2:

H Ham. func. for (1) and $x(t)$ solves (1)

$\Longrightarrow H(x(t)) = $ const.

$$\left[\frac{d}{dt} H(x(t)) = \underset{-f_2}{\underset{\shortparallel}{\frac{\partial H}{\partial x_1}}} \underset{f_1}{\underset{\shortparallel}{\dot{x}_1}} + \underset{f_1}{\underset{\shortparallel}{\frac{\partial H}{\partial x_2}}} \underset{f_2}{\underset{\shortparallel}{\dot{x}_2}} = 0 \right]$$

Obs. 2:

i) $H(x(t))$ is <u>conserved</u>.

ii) Level curves of H = phase trajectories of (1)

iii) x_0 equi. pt. of (1)

$$\Updownarrow$$

$$\frac{\partial H}{\partial x_1}(x_0) = -f_2(x_0) = 0 \;,\; \frac{\partial H}{\partial x_2}(x_0) = f_1(x_0) = 0$$

$$\Updownarrow$$

x_0 crit. pt. of H

iv) 2nd deriv. test:

$$q = \det D^2 H = \det \begin{bmatrix} H_{x_1x_1} & H_{x_1x_2} \\ H_{x_2x_1} & H_{x_2x_2} \end{bmatrix}$$

$$= H_{x_1x_1} H_{x_2x_2} - H_{x_1x_2} H_{x_2x_1}$$

$$= \lambda_1(D^2H) \cdot \lambda_2(D^2H) \qquad [\lambda_i \text{ curvature in } r_i \text{ dir'n}]$$

x_0 str. min. $\Rightarrow \lambda_1, \lambda_2 > 0$

$q(x_0) > 0 \Rightarrow x_0$ strict min. or str. max. for H

$q(x_0) < 0 \Rightarrow x_0$ saddle pt. for H

$q(x_0) = 0 \Rightarrow$ no conclusion

v) Near str. min./max. x_0 of H,

level curves (=ph. trajectories) are closed curves

$\Rightarrow x_0$ <u>center</u> for (1)

vi) Saddle for $H \Rightarrow$ saddle for (1) [linearization!]

We have shown:

<u>Lem. 3:</u>

Assume (1) Ham. syst. and $f(x_0)=0$.

a) $q(x_0) > 0 \Rightarrow x_0$ center

b) $q(x_0) < 0 \Rightarrow x_0$ saddle pt.

c) $q(x_0) = 0 \Rightarrow$ no conclusion

(higher order equi. pt.)

<u>Ex. 1:</u> (cont.)

Equi. pt's for (*): (chk.)

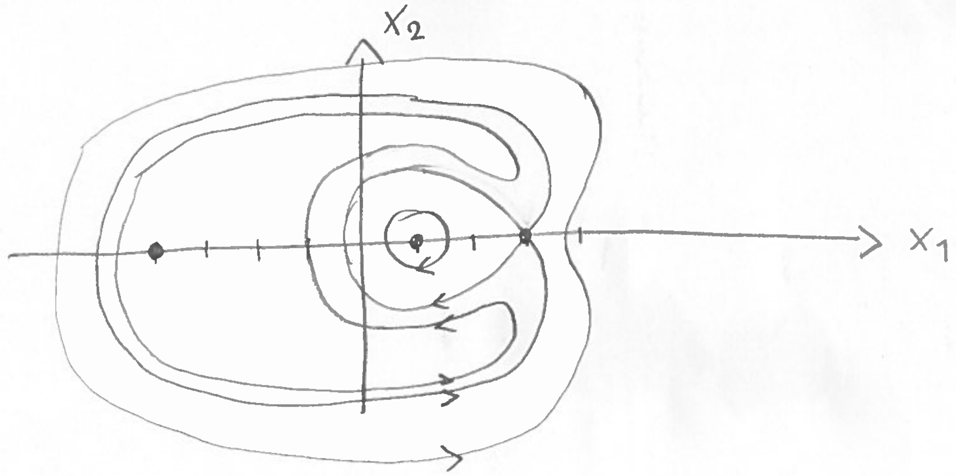
$(1,0), (3,0), (-4,0)$ [x_2 must be 0!]

$$q = \det D^2 H = H_{x_1x_1} H_{x_2x_2} - H_{x_1x_2} H_{x_2x_1}$$

$$\overset{(*)\text{ Ham.}}{=} \left(-\frac{\partial f_2}{\partial x_1}\right)\frac{\partial f_1}{\partial x_2} - \left(-\frac{\partial f_2}{\partial x_2}\right)\left(\frac{\partial f_1}{\partial x_1}\right) \quad \left[f_1 = \frac{\partial H}{\partial x_2},\ f_2 = -\frac{\partial H}{\partial x_1}\right]$$

$$\overset{\text{chk.}}{=} (13 - 3x_1^2 - x_2^2)(13 - x_1^2 - 3x_2^2) - 4x_1^2 x_2^2$$

	$(1,0)$	$(3,0)$	$(-4,0)$
q	120	-56	105
type	center	saddle	center



Obs. 3:

i) Lem. 3 can show center for <u>non</u>-lin. (Ham.) syst.
(lin. can not!)

ii) Conclusion of Lem. 3 = cond. for linearized syst. (chk)

iii) Ham. syst. have no $\underbrace{\text{nodes/spirals}}_{\text{no conservation}}$!

iv) All above also holds for $2n \times 2n$ Ham. syst.:

$$\dot{x}_1 = \nabla_{x_2} H(x_1, x_2) \quad , \quad \dot{x}_2 = -\nabla_{x_1} H(x_1, x_2)$$

<u>But</u>: Lem 3 gives center when all $\lambda_i(D^2H) < 0$ (or >0) etc.
(can not look at φ anymore)

v) Conservative syst. are Ham.:

$$\ddot{x} = -\nabla \varphi(x) \iff \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\nabla \varphi(x_1) \end{cases} \implies H(x_1, x_2) = \frac{1}{2} x_2^2 + \varphi(x_1)$$

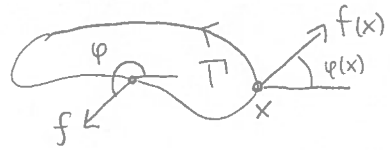
Ham. funk = total energy!

B. Index theory

Assume:

1) Γ simple, closed, p.w. reg. C^1 curve, oriented counter clk.wise

2) $f \in C^1(\mathbb{R}^2;\mathbb{R}^2)$; $f(x) \neq 0$ on Γ (no equi. pt. there)



Def. 2:

a) Polar angle of f at x : $\varphi(x) = \arctan \frac{f_2(x)}{f_1(x)}$

b) Index of Γ : $\boxed{I_\Gamma = \frac{1}{2\pi} \oint_\Gamma d\varphi}$

Rem. 1:

i) Γ reg. C^1 $\Rightarrow$ exists C^1 param. $y(s)$ s.t. $\dot{y} \neq 0$

ii) $y(s)$, $s \in [s_0, s_1]$, reg. C^1 param. for Γ

$$\Rightarrow \oint_\Gamma d\varphi = \int_{s_0}^{s_1} \frac{d}{ds}\Big(\varphi(y(s))\Big)\, ds$$

$$\overset{\text{chk.}}{=} \int_{s_0}^{s_1} \frac{f_1(y(s))\frac{d}{ds}\big(f_2(y(s))\big) - f_2(y(s))\frac{d}{ds}\big(f_1(y(s))\big)}{f_1(y(s))^2 + f_2(y(s))^2}\, ds$$

Ex. 2:

$$f(x) = \begin{bmatrix} 2x_1^2 - 1 \\ 2x_1x_2 \end{bmatrix}$$

Γ : $y(s) = (\cos s, \sin s)$, $s \in [0, 2\pi]$ (circle)

Chk.: $f(y(s)) = \begin{pmatrix} \cos 2s \\ \sin 2s \end{pmatrix}$

$$I_\Gamma = \frac{1}{2\pi}\int_0^{2\pi} \frac{f_1 \frac{d}{ds} f_2 - f_2 \frac{d}{ds} f_1}{f_1^2 + f_2^2}\, ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(2s)\, 2\cos 2s \; - \sin 2s \cdot (-2\sin 2s)}{1}\, ds$$

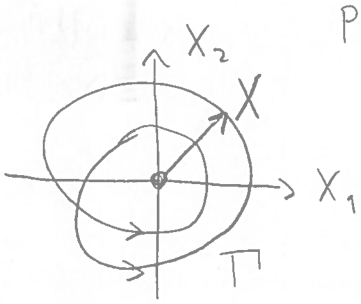
$$= \frac{1}{2\pi}\int_0^{2\pi} 2\, ds = \underline{\underline{2}}$$

Def. 3:

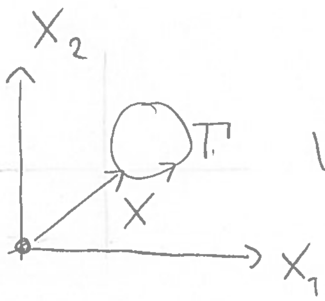
Γ ~~satisfy assumption 1)~~ closed, oriented counter clockwise, $X(s)$, $s \in [s_0, s_1]$ reg. C^1 param, then winding number of Γ about 0 is

$$W_\Gamma = \frac{1}{2\pi}\oint_\Gamma d\varphi_X = \frac{1}{2\pi}\oint_\Gamma d\left(\arctan \frac{X_2}{X_1}\right) = \frac{1}{2\pi}\int_{s_0}^{s_1} \frac{X_1 X_2' - X_2 X_1'}{X_1^2 + X_2^2}\, ds$$

Polar angle of X



$W_\Gamma = 2$



$W_\Gamma = 0$

Obs. 4:

i) $W_\Gamma = \frac{1}{2\pi}\oint_\Gamma \nabla\varphi_X \cdot dX = \frac{1}{2\pi}\int_{X(s_0)}^{X(s_1)} \nabla\varphi_X(x)\cdot dx = \frac{\varphi_X(X(s_1)) - \varphi(X(s_0))}{2\pi}$

$\Longrightarrow \boxed{W_\Gamma \in \mathbb{Z}}$ since $X(s_1) = X(s_0)$ $\left(\Rightarrow \varphi_X(X(s_1)) = \varphi(X(s_0)) + k\cdot 2\pi\right)$

ii) $\Gamma_f = \{ f(x) : x \in \Gamma \}$

$\Longrightarrow I_\Gamma = W_{\Gamma_f}$ $\quad [X(s) = f(y(s))]$

iii) I_Γ = no. times Γ_f winds about 0 counter clockwisely

Therefore (chk!):

Lem. 4:

a) $I_\Pi \in \mathbb{Z}$

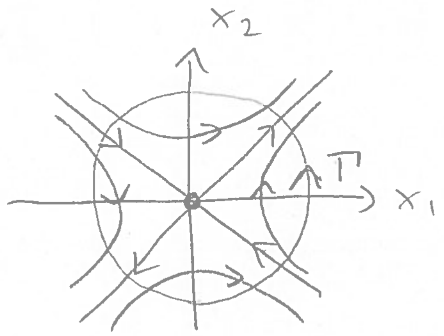
b) $I_\Pi = \frac{1}{2}(p-q)$ where

p = no. of times f crosses a given axis counter clockwisely during one revolution of Π

q = ——— " ———— clockwisely ——— " ————

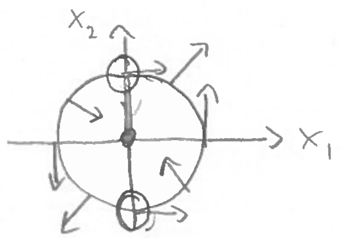
Ex. 3: Saddle

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 \end{cases}$$



$\left(\lambda = \pm 1 \quad r = \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} \right)$

$\Pi : x_1^2 + x_2^2 = 1$



x_1-axis crossed by f (by Π_f)

$2\times$ clockwise

$0\times$ c. —"—

$$\Longrightarrow I_\Pi = \frac{1}{2}(p-q) = \frac{1}{2}(0-2) = \underline{\underline{-1}}$$