Diff. likn. 12.2.2013

- Ref. gr. fredag
- Do HW prob's

A. The matrix exponential

Last time:

$$A \in \mathbb{C}^{n\times n} \Rightarrow e^{A} := \lim_{n\to\infty} \left(I + A + \dots + \frac{1}{n!}A^{n}\right)$$

If $\Phi(t) = e^{tA}$, $t \in \mathbb{R}$, then

(1) $\dot{\Phi} = A\Phi$ in $\mathbb{R}$, $\Phi(0) = I$

Lem. 1:

a) e^{tA} fund. matr.

b) $e^{(t+s)A} = e^{tA} \cdot e^{sA}$

Pf.:

a) By (1) and $\Phi(0)$ inv. $\Rightarrow$ $\Phi(t)$ inv. $t \in \mathbb{R}$. (previously)

b) Lem. 2

$\square$

<u>Lem. 2:</u> $A, B \in \mathbb{C}^{n\times n}$, $A\cdot B = B\cdot A$

a) $e^{A}\cdot B = B\cdot e^{A}$

b) $e^{A+B} = e^{A}\cdot e^{B} = e^{B}\cdot e^{A}$

<u>Pf.:</u> HW 5, ideas for b):

$\Psi(t) := e^{t(A+B)} - e^{tA}\cdot e^{tB}$

Chk. using a): $\dot{\Psi} = (A+B)\Psi$, $\Psi(0)=0$

Uniqueness: $\Psi(t) \equiv 0$ in $\mathbb{R}$ (take $t=1$) $\square$

<u>Lem. 3:</u>

a) $(e^{A})^{-1} = e^{-A}$

b) $A = PJP^{-1} \Rightarrow e^{A} = Pe^{J}P^{-1}$

<u>Pf.:</u>

a) $A\cdot(-A) = (-A)\cdot A \overset{\text{Lem. 2 b)}}{\Longrightarrow} I = e^{A-A} = e^{A}\cdot e^{-A}$

b) $A^{n} = PJP^{-1}PJP^{-1}\cdots PJP = PJ^{n}P^{-1}$

$$\Rightarrow \underbrace{I + \dots + \frac{1}{n!}A^{n}}_{\downarrow \atop e^{A}} = P\Big(\underbrace{I + \dots + \frac{1}{n!}J^{n}}_{\downarrow \atop e^{J}}\Big)P^{-1} \qquad n\to\infty$$

$\square$

Hence, w. $\Phi(t) = e^{tA}$,

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t}\Phi(s)^{-1}b(s)\,ds$$
$$\boxed{= e^{(t-t_0)A}x_0 + \int_{t_0}^{t} e^{(t-s)A}b(s)\,ds}$$

is the sol'n of

(2) $\dot{x} = Ax + b(t)\;;\; x(t_0) = x_0 \quad (A \in \mathbb{R}^{n\times n})$

Ex. 1:

$$A = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix} \overset{\text{chk}}{=} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = P\Lambda P^{-1}$$

$$e^{tA} \overset{\text{Lem.3}}{=} P e^{t\Lambda} P^{-1} \overset{\text{last time}}{=} P \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} P^{-1}$$

$$\div \quad \Big| = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -e^{-t} \\ e^{-2t} & e^{-2t} \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} e^{-2t} & e^{-2t} - e^{-t} \\ 0 & e^{-t} \end{bmatrix}}}$$

Ex. 2: (HW 5)

$$A = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_m \end{bmatrix} \quad \text{block-diag}$$

Chk: $A^n = \begin{bmatrix} B_1^n & & 0 \\ & \ddots & \\ 0 & & B_m^n \end{bmatrix}$

$$S_n = I + \cdots + \frac{1}{n!}A^n = \begin{bmatrix} \sum_{k=0}^{n}\frac{1}{k!}B_1^n & & 0 \\ & \ddots & \\ 0 & & \sum_{k=0}^{n}\frac{1}{k!}B_m^n \end{bmatrix}$$

$$\overset{n\to\infty}{\Longrightarrow} e^A = \begin{bmatrix} e^{B_1} & & 0 \\ & \ddots & \\ 0 & & e^{B_m} \end{bmatrix}$$

HW 5: More ex.!

B. Stability of sol'ns

(3) $\dot{x} = A(t)x + b(t)$; $A : \mathbb{R} \to \mathbb{R}^{n \times n}$; $b, x : \mathbb{R} \to \mathbb{R}^n$

Recall:

i) Sol'n $x(t)$ stable if for all $\varepsilon > 0$, exist $\delta > 0$, s.t.

$$|x(t_0) - a| < \delta \Rightarrow |x(t) - \varphi(t; a, t_0)| < \varepsilon \text{ for all } t \geq t_0.$$

ii) There exists fund. matr. Φ of A (A cont.!) and

$$x(t) = \Phi(t)\Phi(t_0)^{-1}x(t_0) + \Phi(t)\int_{t_0}^{t} \Phi(s)^{-1} b(s)\,ds$$

iii) $$|x(t) - \varphi(t; a, t_0)| \overset{\text{chk.}}{=} |\Phi(t)\Phi^{-1}(t_0)(x(t_0) - a)|$$

$$\leq \|\Phi(t)\| \cdot \|\Phi^{-1}(t_0)\| \cdot |x(t_0) - a|$$

$\boxed{\text{Hence}}$, as in 2×2 case:

Thm. 1: Φ fund. matr. for $A(t)$

a) $\left.\begin{array}{l}\|\Phi(t)\| \leq M < \infty \\ \text{for all } t \geq t_0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{all sol'ns of (3)} \\ \underline{\text{stable}}\ (t \geq t_0)\end{array}\right.$

b) $\lim_{t\to\infty} \|\Phi(t)\| = 0 \Rightarrow$ all sol'ns of (3) $\underline{\text{asympt. stab.}}$ ($t \geq t_0$)

c) $\lim_{t\to\infty} \|\Phi(t)\| = \infty \Rightarrow$ all sol'ns of (3) $\underline{\text{unstab.}}$ ($t \geq t_0$)

<u>Pf:</u>

iii) $\Rightarrow$ a) and b)

c) follows as in 2×2 case $\square$

<u>Rem. 7:</u>

$\|\Phi\| \leq M$, $t \geq t_0$ $\Leftrightarrow$ all sol'ns of (3) w. $b \equiv 0$ are bnd.

$\|\Phi\| \to \infty$, $t \to \infty$ $\Leftrightarrow$ at least one sol'n of (3) w. $b \equiv 0$ $\to \pm\infty$ as $t \to \infty$.

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C. <u>Stability when $A \in \mathbb{R}^{n\times n}$ const.</u>

From Thm. 7 as in 2×2 case:

<u>Thm. 2:</u> $\lambda_1, \dots, \lambda_n$ eig. val's of $A \in \mathbb{R}^{n\times n}$

i) All sol'ns of (3) <u>stab.</u> $\Rightarrow$ $\max_j \operatorname{Re}\lambda_j \leq 0$

ii) $\max_j \operatorname{Re}\lambda_j \leq 0$ and $\lambda_i \neq \lambda_j$ for $i \neq j$ $\Rightarrow$ all sol'ns of (3) <u>stab.</u>

iii) $\max_j \operatorname{Re}\lambda_j < 0$ $\Rightarrow$ all sol'ns of (3) <u>asympt. stab.</u>

iv) $\max_j \operatorname{Re}\lambda_j > 0$ $\Rightarrow$ — " — <u>unstab.</u>

<u>Pf.:</u>

ii) $\lambda_i \neq \lambda_j$, $i \neq j$ $\Rightarrow$ lin. indep. eig. vec's $r_1, \dots, r_n$

$\Rightarrow$ Basis $(b\equiv 0)$: $|x_j| = |r_j||e^{\lambda_j t}| = |r_j| e^{\operatorname{Re}\lambda_j t}$

$\Rightarrow$ Φ bnd., $t \geq t_0$ if $\max \operatorname{Re}\lambda_j \leq 0$

iii) Basis for (3) w. $b \equiv 0$:

$$|x_i| = |p_j(t)|\,|e^{\lambda_j t}| = |p_j(t)|\, e^{\mathrm{Re}\,\lambda_j t}$$

$\uparrow$

polynomial, $|p_j(t)| \underset{t\to\infty}{\longrightarrow} \infty$

$$\Rightarrow \Phi \underset{t\to\infty}{\longrightarrow} 0 \quad \text{if} \quad \max_j \mathrm{Re}\,\lambda_j < 0 \qquad \square$$

Ex. 3:? $\dot{x} = Ax$, $A = \begin{bmatrix} -10 & 2 & 1 \\ 0 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix}$, chk: $\lambda = -10$, $\lambda = \pm 4i$; $\max_j \mathrm{Re}\,\lambda_j = 0 \overset{\text{Thm. 2 ii)}}{\underset{\lambda_i \neq \lambda_j}{\Rightarrow}}$ all sol'ns stable

D. Stability when $A: \mathbb{R} \to \mathbb{R}^{n\times n}$ not const.

Assume:

$$(A)\ \begin{cases} A(t) = B + C(t) \\ \text{where } B \in \mathbb{R}^{n\times n} \text{ const.}, \ \int_{t_0}^{\infty} \|C(t)\|\,dt < \infty \end{cases}$$

Thm. 3: Assume (A), Φ fund. matr. for B ($\dot{\Phi} = B\Phi$)

a) $\left.\begin{array}{l} \|\Phi(t)\| \le M \\ \text{for all } t \ge t_0 \end{array}\right\} \Rightarrow$ all sol'ns of (3) are stab.

b) $\lim_{t\to\infty} \|\Phi(t)\| = 0 \Rightarrow$ all sol'ns of (3) asympt. stab.

From Thm. 3 it follows that:

Thm. 4: Assume (A), $\lambda_1, \ldots, \lambda_n$ eig.val's of B

a) $\max_j \mathrm{Re}\,\lambda_j \le 0$; $\lambda_i \neq \lambda_j$ when $i \neq j$

$\Rightarrow$ all sol'ns of (3) stab.

b) $\max_j \operatorname{Re}\lambda_j < 0 \Rightarrow$ all sol'ns of (3) asympt. stab.

Ex. 3⁴:

$$\ddot{x} + a\dot{x} + (b + ce^{-t}\cos t)x = 0; \quad a, b > 0$$

$$\Updownarrow \quad \begin{array}{l} x_1 = x \\ x_2 = \dot{x}_1 \end{array}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \Big(\underbrace{\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}}_{B} + \underbrace{\begin{bmatrix} 0 & 0 \\ -ce^{-t}\cos t & 0 \end{bmatrix}}_{C(t)} \Big) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Chk.: $\lambda_B = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right) \overset{a,b>0}{\Longrightarrow} \operatorname{Re}\lambda_B < 0$

$$\int_0^\infty \|C(t)\|\,dt = |c| \int_0^\infty e^{-t} \underbrace{|\cos t|}_{\leq 1}\,dt < \infty$$

Thm. $4^{b)} \Rightarrow$ all sol'ns asympt. stab.

E. Pf. of Thm. 3

? Grönwall's lemma:

If $u, v \geq 0$, cont., and for $K > 0$,

$$(*) \qquad u(t) \leq K + \int_{t_0}^t u(s)\,v(s)\,ds, \quad t \geq t_0,$$

then

$$u(t) \leq K e^{\int_{t_0}^t v(s)\,ds}, \quad t \geq t_0$$

8.)

Pf.:

$$\frac{u(t)}{K+\int_{t_0}^{t} uv} \overset{(*)}{\leq} 1 \overset{\cdot v}{\Longrightarrow} \frac{uv}{K+\int_{t_0}^{t} uv} \leq v$$

$$\Longrightarrow \frac{d}{dt} \ln\left(K+\int_{t_0}^{t} uv\right) \leq v$$

$$\overset{\text{int.}}{\Longrightarrow} \ln\left(K+\int_{t_0}^{t} uv\right) - \ln K \leq \int_{t_0}^{t} v$$

$$\overset{\text{exp.}}{\Longrightarrow} \left(u(t) \overset{(*)}{\leq}\right) K + \int uv \leq e^{\int_{t_0}^{t} v + \ln K} = K e^{\int_{t_0}^{t} v\,ds} \qquad \square$$

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Oppg. 1 – φσ 2