Diff. eq'ns 19.3.13

A. Unstable equi. pt.'s

(1) $\dot{x} = f(x)$; $x: \mathbb{R} \to \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^n$

Thm. 1: Assume $f(0)=0$ and f Lip. in domain $B \ni 0$.

Then $x=0$ unstab. equi pt. if exists U s.t.

i) U is C^1

ii) $U(0) = 0$

iii) for all $\delta > 0$ exists x s.t. $|x| < \delta$ and $U(x) > 0$

iv) exists $\eta > 0$ s.t.

$$\dot{U}(x) = \nabla U(x) \cdot f(x) > 0$$

for all x s.t. $|x| < \eta$ and $U(x) > 0$.

Rem. 1:

a) Typical U's: $x_1 x_2$, $x_1^2 - x_2^2$, $\sum_i \sum_{j \neq i} x_i x_j$, ...

b) Same conclusion w. $U, \dot{U} < 0$ in iii) and iv)
[then $-U$ satisfy. i) - iv)]

Ex. 1:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -(1-x_1^2)x_2 + x_1 \end{cases}$$

0 equi. pt.

$U(x_1,x_2) = x_1 \cdot x_2$

i) – iii) $U \in C^1$, $U(0)=0$, $U(s,s)=s^2>0$ for $s\neq 0$

iv) $\dot{U} = \dot{x}_1 x_2 + x_1\dot{x}_2 = x_2^2 - (1-x_1^2)x_2x_1 + x_1^2$

$$\overset{x_1^2<1}{>} x_1^2+x_2^2 - \underbrace{|x_1x_2|}_{\leq \frac{1}{2}(x_1^2+x_2^2)} > \frac{1}{2}(x_1^2+x_2^2) \overset{x_1x_2>0}{>} 0$$

when $U = x_1\cdot x_2 > 0$ (and $x_1^2<1$)

Thm. 1 $\Rightarrow$ 0 unstab.

Pf. of Thm 1:

1) Let $0<\delta<\frac{\eta}{2}$ and x_0 s.t.

$|x_0|<\delta$ and $U(x_0)>0$ (iii)

Def.

$K = \{x : |x| \leq \frac{\eta}{2},\ U(x) \geq U(x_0)\}$

K is closed, bnd. (U cont.)

If x solves (1) and $x(t)\in K$ for all $t\geq t_0$,

then

$$\frac{d}{dt}U(x(t)) \overset{(1)}{\geq} \min_{x\in K} \dot{U}(x) \overset{\dot{U}\text{ cont}}{=:} \gamma \underset{K\text{ closed, bnd}}{\overset{\text{iv), ii)}}{>}} 0$$

for all $t\geq t_0$.

2) Let $x(t) = \varphi(t; x_0, t_0)$ $\quad (x_0 \in K\,!)$

Assume $x(t) \in K$, $t \geq t_0$

$$\overset{1)}{\Longrightarrow} \frac{d}{dt} U(x(t)) \geq \gamma > 0$$

$$\Longrightarrow U(x(t)) \underset{t\to\infty}{\longrightarrow} \infty$$

Contradiction, since K closed, bnd and U cont.

(so $\max_{x\in K} U(x) < \infty$)

Hence $x(t)$ leaves K and then $\{|x| \leq \frac{\eta}{2}\}$

(since $U(x_0) > 0$, $\dot{U} > 0 \Rightarrow U(x(t)) > 0, t > t_0$)

in finite time.

Conclusion: For all $\delta > 0$ exists x_0 s.t.

$$|x - x_0| < \delta \Rightarrow \max_{t \geq t_0} |\varphi(t; x_0, t_0) - 0| > \frac{\eta}{2}$$

and 0 unstab. $\square$

B. Linearization again

Thm. 2: (earlier)

Assume f C^1, $f(x_0) = 0$, λ_i eig-val's of $Df(x_0)$, then

$$\max_i \operatorname{Re} \lambda_i > 0 \Rightarrow x = x_0 \text{ unstab. equi. pt. of } (1).$$

Lem. 1: If $A \in \mathbb{R}^{n\times n}$ has eig. val's λ_i s.t.

$$\alpha \le \mathrm{Re}\,\lambda_i \le \beta \qquad (\alpha, \beta \in \mathbb{R}),$$

then for all $\varepsilon > 0$ exists $\tilde{A}$ similar to A s.t.

$$(\alpha - \varepsilon)|x|^2 \le x^T \tilde{A} x \le (\beta + \varepsilon)|x|^2.$$

Rem. 2:

Can take $\varepsilon = 0$ when A has n lin. indep. eig. vec's!

Pf. of Thm. 2:

1) $x_1, x_2 \equiv x_0$ solves (1)

$\Rightarrow$ $Py = x_1 - x_2$ solves

$$(*) \qquad \dot{y} = \tilde{A} y + o(|y|) \text{ as } y \to 0$$

where $\tilde{A} = P^{-1} Df(x_0) P$ can be chosen on Jordan form

$$\tilde{A} = \begin{bmatrix} B_1 & 0 \\ 0 & \bar{B} \end{bmatrix}$$

s.t. B_1 corresponds to eig. val. λ_j w. max. Re-value

2) Let $\beta = \max_i \mathrm{Re}\,\lambda_i = \mathrm{Re}\,\lambda_j$, $\bar{\beta} = \max_{i \ne j} \mathrm{Re}\,\lambda_i < \beta$.

By Lem. 1, can take P s.t.

$$(\beta - \varepsilon)|y_1|^2 \le y_1^T B_1 y_1, \quad \bar{y}^T \bar{B} \bar{y} \le (\bar{\beta} + \varepsilon)|\bar{y}|^2$$

where $y = (y_1, \bar{y})$ and $0 < \varepsilon < \frac{1}{2}(\beta - \bar{\beta})$.

3) Def. $\mathcal{U}(y_1, \bar{y}) = \frac{1}{2}(|y_1|^2 - |\bar{y}|^2)$

Obs: $\mathcal{U}$ satisfy i), ii), iii) of Thm. 7

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$$\frac{d}{dt}\mathcal{U}(y(t)) = y_1 \dot{y}_1 - \bar{y}\cdot\dot{\bar{y}}$$

$$\overset{(*)}{=} y_1^T B_1 y_1 - \bar{y}^T \bar{B}\bar{y} + o(|y|^2)$$

$$\overset{2)}{\geq} (\beta - \varepsilon)|y_1|^2 - (\bar{\beta}+\varepsilon)|\bar{y}|^2 + o(|y|^2)$$

$$\mathcal{U}(y) > 0 \Rightarrow |y_1| > |\bar{y}| \geq 0$$

$$\Rightarrow \frac{d}{dt}\mathcal{U}(y(t)) \geq \underbrace{(\beta - \bar{\beta} - 2\varepsilon)}_{>0}|y_1|^2 + o(|y|^2) > 0$$

for $|y|$ small enough.

Conclusion: $\mathcal{U}$ satisfies i) – iv) of Thm. 7

$\Rightarrow$ $y=0$ $(x = x_0)$ unstab. equi.pt. for $(*)$ (for (7)) $\square$

Ex. 2:

$$(**)_\pm \begin{cases} \dot{x}_1 = x_2 \pm x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1 \pm x_2(x_1^2 + x_2^2) \end{cases}$$

$$Df(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \lambda_\pm = \pm i, \quad \max_\pm \operatorname{Re}\lambda_\pm = 0$$

$\Rightarrow$ no conclusion by linearization

6.)

$$W = \frac{1}{2}(x_1^2 + x_2^2)$$

$$\dot{W} = x_1 \dot{x}_1 + x_2 \dot{x}_2 \overset{(**)_\pm}{=} \pm (x_1^2 + x_2^2)^2$$

$V = W$ str. L. func. for $(**)_-$ $\Rightarrow$ 0 asympt. stab. in $(**)_-$

$U = W$ satisfy Thm. 7 for $(**)_+$ $\Rightarrow$ 0 unstab. in $(**)_+$

Obs. 1:

Lin. syst. stab. (not asympt. stab.) (center at $x=0$), but non-lin. syst. either asympt. stab. or unstab. !

C. Proof of Lem. 1

1) $A = B_i$ = Jordan block of λ_i

a) $B_i = \lambda_i \in \mathbb{R} \Rightarrow x^T B_i x = \lambda_i |x|^2 \quad (x \in \mathbb{R}^1)$

b) $B_i = \lambda_i I + N = \begin{bmatrix} \lambda_i & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda_i \end{bmatrix} \in \mathbb{R}^{d \times d}$

$D_\varepsilon = \begin{bmatrix} 1 & & 0 \\ & \varepsilon & \\ 0 & & \varepsilon^{d-1} \end{bmatrix}, \quad D_\varepsilon^{-1} B_i D_\varepsilon =$

$$D_\varepsilon^{-1} B_i D_\varepsilon = \begin{bmatrix} 1 & & 0 \\ & \frac{1}{\varepsilon} & \\ 0 & & \frac{1}{\varepsilon^{d-1}} \end{bmatrix} \cdot \begin{bmatrix} \lambda_i & \varepsilon & & 0 \\ & \varepsilon\lambda_i & \varepsilon^2 & \\ & & \ddots & \varepsilon^{d-1} \\ 0 & & & \varepsilon^{d-1}\lambda_i \end{bmatrix} = \begin{bmatrix} \lambda_i & \varepsilon & 0 \\ & \ddots & \varepsilon \\ 0 & & \lambda_i \end{bmatrix}$$

$$= \lambda_i I + \varepsilon N$$

Obs: $|x^T N x| = |x_1 x_2 + x_2 x_3 + \dots + x_{d-1} x_d|$

$$\le \tfrac{1}{2}(x_1^2 + x_2^2) + \dots + \tfrac{1}{2}(x_{d-1}^2 + x_d^2)$$

$$\le x_1^2 + \dots + x_d^2 = |x|^2,$$

Hence

$$(\lambda_i - \varepsilon)|x|^2 \le \underbrace{x^T \lambda_i x + \varepsilon x^T N x}_{x^T B_i x}$$

$$\le (\lambda_i + \varepsilon)|x|^2$$

c) $B_i = D_i = \begin{bmatrix} \mathrm{Re}\,\lambda_i & \mathrm{Im}\,\lambda_i \\ -\mathrm{Im}\,\lambda_i & \mathrm{Re}\,\lambda_i \end{bmatrix}$

$$\Rightarrow x^T B_i x \underset{\underset{x^T(B-B^T)x=0}{\Uparrow}}{\overset{\text{chk}}{=}} x^T \tfrac{1}{2}(B_i + B_i^T)x = \mathrm{Re}\,\lambda_i |x|^2 \quad (x \in \mathbb{R}^2)$$

d) $B_i = \begin{bmatrix} D_i & I & & 0 \\ & \ddots & \ddots & \\ & & \ddots & I \\ 0 & & & D_i \end{bmatrix}, \quad D_\varepsilon = \begin{bmatrix} I & & & 0 \\ & \varepsilon I & & \\ & & \ddots & \\ 0 & & & \varepsilon^{m-1} I \end{bmatrix}$

$$\overset{\text{chk}}{\rightsquigarrow} (\mathrm{Re}\,\lambda_i - \varepsilon)|x|^2 \le x^T D_\varepsilon^{-1} B_i D_\varepsilon x \le (\mathrm{Re}\,\lambda_i + \varepsilon)|x|^2$$

2) General $A \in \mathbb{R}^{n \times n}$:

A similar w. Jordan form $J = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_m \end{bmatrix}$

$$x^T J x = \sum x_j^T B_j x_j; \quad x = (\underset{\nwarrow}{x_1}, \dots, \underset{\nearrow}{x_m})$$
vectors

where B_j Jordan blocks as in 1).

By 7), exists P_j and $\tilde{B}_j = P_j B_j P_j^{-1}$ s.t.

$$(\alpha - \varepsilon)|x_j|^2 \le (\mathrm{Re}\,\lambda_i - \varepsilon)|x_j|^2$$
$$\le x_j^T B_j x_j$$
$$\le (\mathrm{Re}\,\lambda_i + \varepsilon)|x_j|^2 \le (\beta + \varepsilon)|x_j|^2$$

$$\Rightarrow (\alpha - \varepsilon)|x|^2 = (\alpha - \varepsilon)\sum_j |x_j|^2$$
$$\le \sum x_j^T B_j x_j = x^T \tilde{A} x$$
$$\le (\beta + \varepsilon)\sum_j |x_j|^2 = (\beta + \varepsilon)|x|^2$$

where $\tilde{A} = \begin{bmatrix} \tilde{B}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{B}_m \end{bmatrix} = PJP^{-1}$, $P = \begin{bmatrix} P_1 & & 0 \\ & \ddots & \\ 0 & & P_m \end{bmatrix}$ $\square$