# Diff. eqn's 21.2.13

Messages

- Copy of B-R sold at IMF office
- Read B-R chp. 6.5

## A. Cont. dep. on $f$

(1) $\dot{x} = f(x,t)$

(2) $x(t_0) = x_0$

where $f = (f_1, \dots, f_n)$, $x = (x_1, \dots, x_n)$.

**Thm. 1:** (cont. dep. on data)

Let $R \subset \mathbb{R}^{n+1}$ be a domain. If

i) $f$ cont. and $x$-Lip. on $R$.

ii) $g$ cont. and, there is $\varepsilon > 0$ s.t.

$$|f(x,t) - g(x,t)| \leq \varepsilon \text{ in } R.$$

iii) $(x(t),t), (y(t),t) \in R$ for $t \in [t_0, b)$.

iv) $x, y$ solve

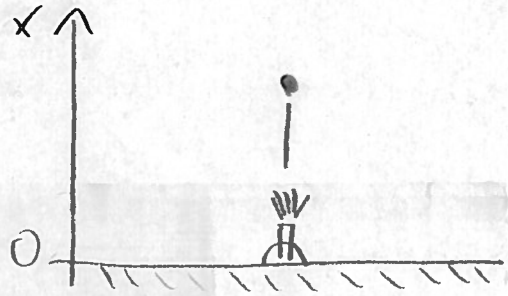
$$\begin{cases} \dot{x} = f(x,t) \text{ in } (t_0,b) \\ x(t_0) = x_0 \end{cases} \text{ and } \begin{cases} \dot{y} = g(y,t) \text{ in } (t_0,b) \\ y(t_0) = y_0 \end{cases},$$

then

$$|x(t)-y(t)| \le e^{2L(t-t_0)}|x_0-y_0| + \frac{\varepsilon}{L}\sqrt{e^{2L(t-t_0)}-1}\,, \quad t\in[t_0,b)$$

where $L \le \max\limits_{(x,t)\in R} \|Df\| < \infty$.

## Ex. 1: Projectile




$$m\frac{d^2x^*}{dt^{*2}} = -mg \qquad \text{(const. g)}$$

$$m\frac{d^2x^*}{dt^{*2}} = -\frac{mg}{\left(1+\frac{x}{R}\right)^2} \qquad \text{(Newton)}$$

$g=\frac{GM}{R^2}$, $R$ planet radius

Scaled eq'ns: $\left(x^* = \overset{\text{max. height}}{X}x,\ t^* = \sqrt{\tfrac{X}{g}}\,t\right)$

$$\ddot{x} = -1, \quad \ddot{x}^\varepsilon = -\frac{1}{(1+\varepsilon x^\varepsilon)^2}; \quad 0<\varepsilon=\frac{X}{R}\ll 1 \text{ (typically)}$$

$$\Updownarrow$$

$$\underbrace{\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = -1\end{cases}}_{f(x)}, \quad \underbrace{\begin{cases}\dot{x}_1^\varepsilon = x_2^\varepsilon\\ \dot{x}_2^\varepsilon = -\dfrac{1}{(1+\varepsilon x_1^\varepsilon)^2}\end{cases}}_{f_\varepsilon(x)}$$

Obs: $$|f(x)-f_\varepsilon(x)| = \sqrt{|\underbrace{x_2-x_2}_{=0}|^2 + \Big|\underbrace{-1+\frac{1}{(1+\varepsilon x_1)^2}}_{f_2-f_2^\varepsilon}\Big|^2}$$

chk.

$$= \varepsilon|x_1|\frac{|2+\varepsilon x_1|}{(1+\varepsilon x_1)^2} \le \varepsilon|x_1|\,2 \le 2\varepsilon X$$

when $|x_1|\le X$ (max. height)

Obs: $Df = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \|Df\| = 1 = L$

(do <u>not</u> need $Df_\varepsilon$ !)

Init. cond'ns (think!)

$x_1(0) = 0 = x_1^\varepsilon(0)$, $\quad x_2(0) = \dot{x}_1(0) = v = \dot{x}_1^\varepsilon(0) = x_2^\varepsilon(0)$

By Thm. 1:

$$|x(t) - x^\varepsilon(t)| \le 2X\varepsilon \cdot \sqrt{e^{2t} - 1} \qquad (|x_1| \le H)$$

Obs: $x_\varepsilon(t) \longrightarrow x(t)$ (unif. on closed int's) as $\varepsilon \to 0$ !

<u>Cor. 1:</u>

Let $R \subset \mathbb{R}^{n+1}$ be a domain. ~~Assume:~~ If

i) $f$ cont. and $x$-Lip. on $R$

ii) $f_\varepsilon$ cont. on $R$ and

$$\max_{(x,t) \in R} |f(t,x) - f_\varepsilon(t,x)| \xrightarrow[\varepsilon \to 0]{} 0,$$

iv) $(x(t), t), (x_\varepsilon(t), t) \in R$ for $\varepsilon > 0, t \in [t_0, b)$,

iii) $|x_0 - x_0^\varepsilon| \xrightarrow[\varepsilon \to 0]{} 0$

v) $x$ and $y$ solve

$$\begin{cases} \dot{x} = f(x,t) \text{ in } (t_0, b) \\ x(t_0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}^\varepsilon = f^\varepsilon(x,t) \text{ on } (t_0, b) \\ x^\varepsilon(t_0) = x_0^\varepsilon, \end{cases}$$

then $x_\varepsilon(t) \xrightarrow[\varepsilon \to 0]{} x(t)$, unif. on int's $[t_0, b_1]$ for $b_1 < b$.

<u>Pf.:</u> Use Thm. 1. !

## Rem. 1:

Cor. 1 $\Rightarrow$ sol'ns of (1) and (2) dep. cont.

on the data $f$ and $x_0$ when $f$ cont., $x$–Lip.

## B. Pf. of Thm. 1

1) $\sigma(t) = |x(t) - y(t)|^2$

$$\dot\sigma(t) = 2(x-y)\cdot(\dot x - \dot y) = 2(x-y)\cdot(f(x,t) - g(y,t))$$

$$\overset{C-S}{\leq} 2|x-y|\cdot|f(x,t) - g(y,t)|$$

2) $$|f(x,t) \underbrace{- f(y,t) + f(y,t)}_{=0} - g(y,t)|$$

$$\leq \underbrace{|f(x,t) - f(y,t)|}_{\leq \text{ i)}} + \underbrace{|f(y,t) - g(y,t)|}_{\text{ii)}}$$

$$\overset{\text{i), ii)}}{\leq} L|x-y| + \varepsilon$$

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3) By 1) and 2)

$$\dot\sigma \leq 2L\underbrace{|x-y|^2}_{=\sigma} + 2|x-y|\varepsilon$$

Obs: $0 \leq (a-b)^2 = a^2 - 2ab + b^2 \Rightarrow 2ab \leq a^2 + b^2$

Let $a = \sqrt{L}|x-y|$, $b = \frac{1}{\sqrt{L}}\varepsilon$, then

$$2|x-y|\cdot\varepsilon = 2a\cdot b \leq a^2 + b^2 = L|x-y|^2 + \frac{1}{L}\varepsilon^2$$

Hence

$$\dot\sigma \leq 3L\sigma + \frac{1}{L}\varepsilon^2$$

4) Int. factor $e^{-3Lt}$

$$\frac{d}{dt}\left(e^{-3Lt}\sigma\right) = e^{-3Lt}\left(\dot{\sigma} - 3L\sigma\right) \overset{3)}{\leq} e^{-3Lt}\,\frac{\varepsilon^2}{L}$$

$$\Downarrow \int_{t_0}^{t}$$

$$e^{-3Lt}\sigma(t) - e^{-3Lt_0}\sigma(t_0) \leq \frac{\varepsilon^2}{L}\int_{t_0}^{t} e^{-3Ls}\,ds$$

$$\Downarrow$$

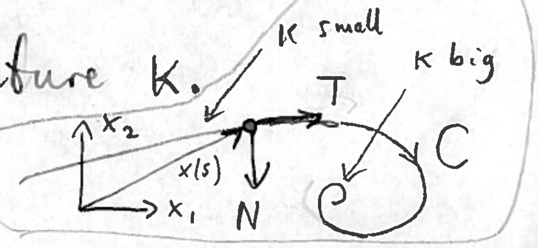
$$\sigma(t) \leq e^{3L(t-t_0)}\sigma(t_0) + \frac{\varepsilon^2}{3L^2}\left(e^{3L(t-t_0)} - 1\right)$$

5) $\sqrt{\sigma} \leq \ldots$ use $\sqrt{|a|+|b|} \leq \sqrt{|a|} + \sqrt{|b|}$ (chk)

(obs: $\frac{3}{2} \leq 2$, $\frac{1}{\sqrt{3}} \leq 1$) $\square$

## C. Application: Regular curves (geometry)

Thm. 2: A regular $C^3$-curve $C$ in $\mathbb{R}^2$ is uniquely determined (up to translation and rotatation) by its curvature $\kappa$.




Regular $C^3$-curve $C$:

There is arclength param. $x(s) \in C^3$ s.t. $\dot{x} \neq 0$; $\dot{x}, \ddot{x}$ lin. indep.
and

$$C = \{x(s) : s \in I\}$$

$\div$ | Lem. 1: $u \in C^1(I;\mathbb{R}^2)$, $|u| = 1 \Longrightarrow 0 = \left(\frac{d}{dt}\frac{1}{2}|u|^2 =\right) u \cdot \frac{du}{dt}$

$\Delta$ | By Obs. 1, x

<u>Recall:</u> (calculus)

a) By def. $\left|\frac{dx}{ds}\right| = 1$ (speed $= 1$)

b) $\vec{T} = \frac{dx}{ds}$ , $\vec{N} = \dfrac{\frac{d\vec{T}}{ds}}{\left|\frac{d\vec{T}}{ds}\right|}$ , $K(s) = \left|\frac{d\vec{T}}{ds}\right|$

$K(s) \neq 0$ ; $0 \neq \ddot{x} = \kappa\dot{T}$ ... $\dot{T} = \ddot{x}$

unit tangent, u. normal, since $\dot{x}, \ddot{x}$ lin. indep.

c) $|\vec{T}| = 1 = |\vec{N}|$ , $\vec{T} \cdot \vec{N} = 0$ $\left[ 0 = \frac{d}{ds} \underbrace{\tfrac{1}{2}|\vec{T}|^2}_{=\frac{1}{2}} = \vec{T} \cdot \underbrace{\frac{d\vec{T}}{ds}}_{\parallel\ \vec{N}} \right]$

d) $x(s) = x(0) + \int_0^s \vec{T}(s)\, ds$

$\Longrightarrow$ T determines x (and C) uniquely up to a transl. $x(0)$.

Obs:

$$(3) \begin{cases} \dfrac{d\vec{T}}{ds} \overset{b)}{=} K(s)\vec{N} \\[2ex] \dfrac{d\vec{N}}{ds} = -K(s)\vec{T} \end{cases} \qquad \boxed{\text{Frenet–Seret eq'ns}}$$

$\left[ 0 = \frac{d}{ds}(\vec{T} \cdot \vec{N}) = \underbrace{\frac{d\vec{T}}{ds}}_{K \cdot \vec{N}} \cdot \vec{N} + \vec{T} \cdot \frac{d\vec{N}}{ds} \Rightarrow \frac{d\vec{N}}{ds} \cdot \vec{T} = -K \right.$

$\left. |\vec{N}| = 1 \Rightarrow \frac{d\vec{N}}{ds} \cdot \vec{N} = 0 \quad \text{(see c))} \right]$

<u>Pf. of Thm. 2:</u>

$[\kappa = |\ddot{x}| \in C^0]$, $\ddot{x} \in C^1$

(3) lin. $6\times 6$ syst. ($\Rightarrow$ loc. x-Lip. since $\kappa$ cont.)

~~+K cont. ($x \in C^3$)~~

$\Longrightarrow$ exists unique sol'n of (3) and

$T(0) = T_0$ , $N(0) = N_0$

$\overset{d)}{\Longrightarrow}$ $\underbrace{x(s)}_{\text{uniq.}}$ determined up to $\underbrace{x(0)}_{\text{transl.}}$, $\underbrace{T(0), N(0)}_{\text{"... frame"}}$ $\square$

## Ex. 2:

Circle $C$: $x(s) = x_0 + r\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right)$ radius $r$, center $x_0$


$\dot{x} = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right)$, $|\dot{x}| = 1$ ($\Rightarrow s$ = arc length, $T = \dot{x}$)

$$\kappa(s) = \left|\frac{dT}{ds}\right| = |\ddot{x}| = \frac{1}{r}\sqrt{\cos^2\frac{s}{r} + \sin^2\frac{s}{r}} = \frac{1}{r}$$

$\Longrightarrow$ any circle (in $\mathbb{R}^2$) has $\kappa(s) = \text{const} \neq 0$.

Thm. 2 uniqueness $\Longrightarrow$ all reg. $C^3$ curves in $\mathbb{R}^2$ w. $\kappa = \text{const.} \neq 0$ are (parts of) circles.

**Rem. 3:** Reg. $C^4$-curves in $\mathbb{R}^3$

$$\dot{\vec{T}} = \kappa(s)\vec{N}, \quad \dot{\vec{N}} = -\kappa(s)\vec{T} + \tau(s)\vec{B}, \quad \dot{\vec{B}} = \tau(s)\vec{B} \quad \text{(F-S eq'ns)}$$

$\tau(s)$: torsion (def.) $\quad$ $\vec{B}$: binormal $\vec{B} = \vec{T} \times \vec{N}$

Curves uniquely determined by $\kappa(s) \neq 0$, $\tau(s)$ (up to transl./rot.)

**HW:** Read B-R chp. 6.5

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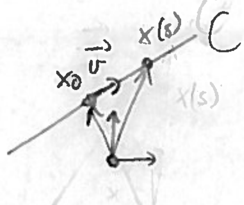
## Ex. 3:

Line $C$: $x(s) = x_0 + s\vec{v}$; $\vec{v}$ const, $|\vec{v}| = 1$.




$|\dot{x}| = |\vec{v}| = 1$ ($\Rightarrow s$ arc length, $T = v$)

$\frac{dT}{ds} = 0 \Rightarrow$ any line has $\kappa(s) \equiv 0$

Obs: Thm. 2 does not apply; $C$ not reg. ($\dot{x}, \ddot{x} = 0$ not lin. indep.)