Diff. eq'ns 20.3.13

A. Invariant and attracting domains

(1) $\dot{x} = f(x)$; $x: \mathbb{R} \to \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^n$

Concepts:

1) Flow: $\varphi(t; x_0)$, sol'n of (1) and $x(0) = x_0$

2) Half trajectory: $T_{x_0}^{\pm} = \{\varphi(t; x_0) : \pm t \geq 0\}$

3) Phase — " — : $T_{x_0} = T_{x_0}^{-} \cup T_{x_0}^{+}$

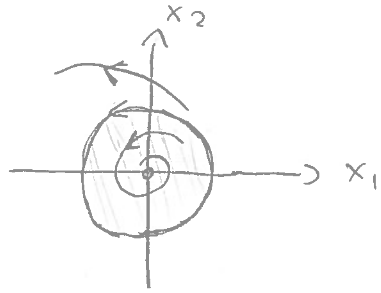
4) $\Omega \subset \mathbb{R}^n$ invariant (positively inv.) under (1):

$$T_{x_0} \subset \Omega \quad (T_{x_0}^{+} \subset \Omega) \text{ for all } x_0 \in \Omega.$$

5) Domain of attraction of equi. pt. x_0:

$$\Omega_a = \{x \in \mathbb{R}^n : \lim_{t \to \infty} \varphi(t; x) = x_0\}$$

Ex. 1:



(2) $\begin{cases} \dot{r} = r(r^2-1) \\ \dot{\theta} = 1 \end{cases}$

(polar coord's)

Equi. pt. $(0,0)$, limit cycle $r^2 = x_1^2 + x_2^2 = 1$ $(\dot{r}=0)$

$0 < r(0) < 1 \Rightarrow \dot{r} < 0$, $\lim_{t\to\infty} r(t) = 0$

$r(0) > 1 \Rightarrow \dot{r} > 1 \Rightarrow r(t) > 1$ for $t \geq 0$.

Domain of attr. of $(0,0)$:

$\Omega_a = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$

Inv. domains:

$\{x : 0 < |x| < 1\}$, $\{x : |x| > 1\}$, $\{0\}$, $\{x : |x| = 1\}$, unions

[$|x| < r$, $|x| > R$; $0 < r < 1 < R$; only pos. inv. !]

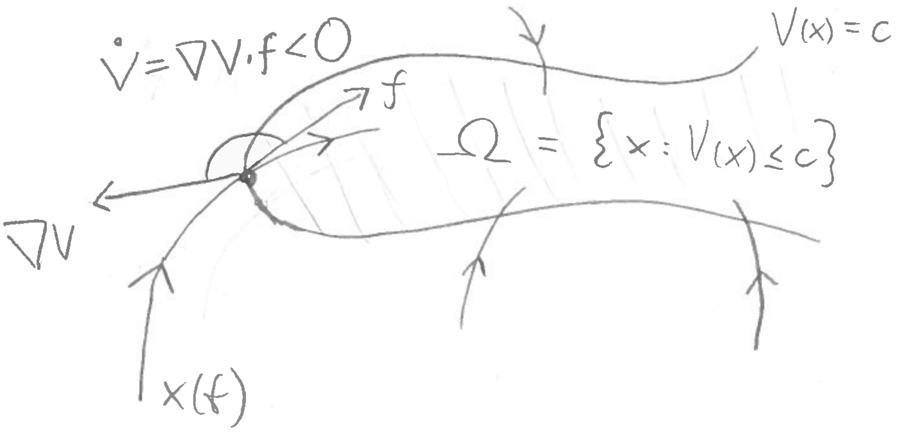
Obs. 1: Closed phase trajectories and equi. pt's are inv.

Lem. 1: Assume $f \in C^1$. If $V \in C^1$ and

$\nabla V(x) \neq 0$ and $\dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0$

for all x s.t. $V(x) = c \in \mathbb{R}$, then

$\Omega = \{x : V(x) \leq c\}$ is pos. inv.



"Pf.:"

Enough to chk. that $V(\varphi(t;x)) \le c$

for $t \in (0, \varepsilon_x)$, $\varepsilon_x > 0$, and x s.t. $V(x) = c$.

(i.e. that $\varphi(t;x)$ do not immediatly exit $V \le c$)

Let x s.t. $V(x) = c$.

1) $\dot V(x) < 0 \implies V(\varphi(t;x)) < c$ for $t \in (0, \varepsilon_x)$

for small enough $\varepsilon_x > 0$

2) $\dot V(x) = 0$:

Since $f \in C^1$, $\varphi(t;y)$ exists and are unique for all y near x and t near 0.

If $\varphi(t;x)$ exits $V \le c$ immediately, then by cont. dep. so must also $\varphi(t;y)$ for all y near x s.t. $V(y) = c$.

By 1), $\dot V(y) = 0$ for these y and hence $V = c$ is part of a phase trajectory there.

But this trajectory crosses $\varphi(t;x)$
and contradicts uniqueness, hence
$V(\varphi(t;x)) \leq c$ for $t \in [0, \varepsilon_x)$, for some $\varepsilon_x > 0$ $\square$

Cor. 1: If $f, V \in C^1$ and

$$\nabla V(x) \neq 0 \text{ and } \dot{V}(x) = \nabla V(x) \cdot f(x) \geq 0$$

for all x s.t. $V(x) = c$, then

$$\Omega = \{x : V(x) \geq c\} \text{ pos. inv.}$$

[$-V(x)$ satisfy Lem. 1]

Ex. 2: (cont.)

$$(2) \overset{\text{chk.}}{\Longleftrightarrow} \begin{cases} \dot{x}_1 = -x_2 - (1 - x_1^2 - x_2^2)x_1 \\ \dot{x}_2 = x_1 - (1 - x_1^2 - x_2^2)x_2 \end{cases}$$

$$V(x_1, x_2) = \tfrac{1}{2}(x_1^2 + x_2^2) \Rightarrow \dot{V} = (x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1)$$

$x_1^2 + x_2^2 \leq c^2$, $c \leq 1$ pos. inv since $\nabla V \neq 0$, $\dot{V} \leq 0$ on $V = \frac{c^2}{2}$

— " — $\geq$ — " — $\geq$ ——— " ——— ≥ 0 — " —

B. Lasalles inv. principle

Thm. 1:

Assume:

i) Ω pos. inv., closed, bnd., $0 \in \Omega$

ii) $f(0) = 0$, f Lip. in Ω

iii) $V(x)$ weak L. func. on Ω

iv) there is no <u>global</u> sol'n $x(t)$ of (1) s.t.

$$x(t) \in \Omega \setminus \{0\},\ t \in \mathbb{R} \quad \text{and} \quad V(x(t)) = c = \text{const.}$$

Then $x = 0$ is <u>asympt. stab.</u> and its <u>domain of attraction</u> contains Ω.

<u>Obs. 2:</u>

i) Weak L-func. can give asympt. stab.!

ii) Thm. 1 gives info about domain of attr. of 0

iii) V strong L func. $\Rightarrow \dot{V} < 0,\ x \neq 0 \Rightarrow$ iv) satisfied

<u>Ex. 3:</u> Van der Pol eq'n

$$(3) \quad \ddot{x} - \beta(x^2 - 1)\dot{x} + x = 0, \quad \beta > 0$$

$$(4) \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \beta(x_1^2 - 1)x_2 - x_1 \end{cases}$$

Only equi. pt.: $(0,0)$

Energy: $E(\dot{x}, x) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2$

$$\left[\,(3)\cdot\dot{x} \iff \frac{d}{dt}\underbrace{\left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2\right)}_{E} = \beta(x^2 - 1)\dot{x}^2\,\right]$$

$V(x_1, x_2) = E(x_1, x_2)$ weak L-func. for $|x_1| \le 1$:

i) – ii) V C^1, $V(0)=0$, $V>0$ for $x\neq 0$, $|x|<1$

iii) $\dot V(x) = x_1\dot x_1 + x_2\dot x_2 = \beta(x_1^2-1)x_2^2 \le 0$ for $|x_1|\le 1$

$\Omega = \{x : V(x)\le \frac{1}{2}\}$ (closed, bnd) pos. inv.

[by Lem. 1 since $\nabla V(x)\neq 0$ and $\dot V\le 0$ on $V=\frac{1}{2}$]

Assume $x(t)$ glob. sol'n s.t. $V(x(t)) = $ const.

$\Rightarrow 0 = \dot V(x(t)) = \beta(x_1^2-1)x_2$

$\Rightarrow x_2 = 0$ or $x_1 = \pm 1$

Hence, if $x_1(t_1)\neq \pm 1 \Rightarrow x_1(t)\neq \pm 1$ in $(-\varepsilon+t_1, t_1+\varepsilon)$
ε small enough

$\Rightarrow x_2(t) = 0$ in $(-\varepsilon+t_1, t_1+\varepsilon)$

$\overset{(4)}{\Rightarrow} \dot x_1(t) = 0$ and $x(t) = $ const ($x_2=0$, $x_1=$ const.)

$\Rightarrow (x_1, x_2) = (0,0)$ (only equi. pt.)

Similarly, if $x_2(t_2)\neq 0 \Rightarrow x_1(t) = \pm 1$

$\overset{(4)}{\Leftarrow\Rightarrow} x_2 = \dot x_1 = 0$ in $(-\varepsilon+t_2, t_2+\varepsilon)$

$\Rightarrow (x_1, x_2) = (0,0)$

Conclusion: i) – iv) of Thm. 1 hold

$\Rightarrow (0,0)$ asympt. stab. and

$\{x : |x|\le 1\}\subset \Omega_a$

Obs. 3:

i) $V(x) = \frac{1}{2}x_1^2 + a x_1 x_2 + \frac{b}{2}x_2^2$ str. L-func. (4)

when e.g. $a = \frac{b}{2} = \frac{2}{3}$ (J-S p. 363)

ii) Chk.: Linearization gives asympt. stab. but no info about Ω_a.

Ex. 4:

$$\dot{x}_1 = -x_1(x_1^2 + x_2^2) + x_2$$
$$\dot{x}_2 = -x_2(x_1^2 + x_2^2) - x_1$$

Chk.: $V = \frac{1}{2}(x_1^2 + x_2^2)$ str. L-func. $\left(\dot{V} = -(x_1^2+x_2^2)^2\right)$

$\Rightarrow B_R = \{x : V(x) \leq \frac{R^2}{2}\}$ pos. inv. (Lem. 1)

Thm. 1 and Obs. 2

$\Rightarrow x = 0$ asympt. stab. and $B_R \subset \Omega_a$ for all $R > 0$

Hence $\Omega_a = \mathbb{R}^2$, $x = 0$ glob. asympt. stab. equi. pt.

C. Pf. of Thm. 1

1) Assume Thm. 1 wrong, then exists sol'n $x(t)$ s.t.

$x(t) \in \Omega$, $t \geq 0$ and $x(t) \underset{t \to \infty}{\not\to} 0$.

2). Since $\Omega \subset \mathbb{R}^n$ closed, bnd. all sequences in Ω have conv. subsequenses (Bolzano-Weierstrass)

Hence $\{x(t_n)\}_n$, $t_n \nearrow \infty$, has subsequence s.t.

$$(\Delta) \qquad x(t_n) \underset{n\to\infty}{\longrightarrow} x_0 \in \Omega$$

By 1) we may assume $x_0 \neq 0$.

[If all conv. subsequences $\to 0$, then $x(t) \to 0$ since Ω bnd.]

3) $\bar{x}(t) = \varphi(t; x_0)$ glob. sol'n in Ω:
($\bar{x}(t)$ exists for $t \in \mathbb{R}$ since f Lip.)

i) Ω pos. inv. $\Rightarrow \bar{x}(t) \in \Omega$ for $t \geq 0$

ii) $\varphi(t; x(t_{n+k}))$ def. for $t \in [-t_n, 0]$, $k \geq 0$

$[\varphi(-t_n; x(t_n)) = \varphi(0; x(0)) = x(0)]$

By cont. dep. of init. data and (Δ),

$$\varphi(t; x(t_{n+k})) \longrightarrow \varphi(t; x_0) \text{ for } t \in [-t_n, 0]$$

Since Ω closed, $\varphi(t; x_0) \in \Omega$, and since $t_n > 0$ arbitrarily big, $\bar{x}(t)$ exists for all $t \leq 0$.

4) $V(\bar{x}(t)) = \text{const.}$:

Let $V(x_0) =: \alpha$. Then $\alpha > 0$ $(x_0 \neq 0)$ and

$$(\square) \qquad \lim_{n\to\infty} V(x(t_n)) = \alpha \qquad (V \text{ cont.} + (\Delta))$$

Let $t_{n_1} \le t_n + s \le t_{n_2}$ $(s \in \mathbb{R})$, then

$$V(x(t_{n_1})) \geq V(x(t_n+s)) \geq V(t_{n_2}) \quad (\dot{V} \le 0)$$

and $V(x(t_n+s)) \xrightarrow[n\to\infty]{} V(x_0)$ by (□).

But $x(t_n+s) = \varphi(s; x(t_n)) \to \varphi(s; x_0) = \bar{x}(s)$

$\overset{V \text{ cont.}}{\Longrightarrow}$ $V(x(t_n+s)) \to V(\bar{x}(s))$

Hence $V(\bar{x}(t)) = \alpha\,(>0)$, $t \in \mathbb{R}$.

3 and 4 contradicts iv), so assumption in 1 is wrong and Thm. 1 holds □