Dynsys 23.1.13

- Info fra www
  - ref. gr.
  - øv. time, hvem går

A. The fundamental matrix

(1) $\dot{\vec{x}} = A\vec{x}$, $A \in \mathbb{R}^{2\times 2}$

(2) $\vec{x}(t_0) = \vec{x}_0$

Last time: There are lin. indep. $\vec{x}_1, \vec{x}_2$ s.t.

all sol'ns of (1) are of the form

$\vec{x} = C_1\vec{x}_1 + C_2\vec{x}_2$; $C_1, C_2 \in \mathbb{R}$.

Obs. 1:

collumns

i) $\vec{x} \overset{\text{chk.}}{=} \underbrace{[\vec{x}_1, \vec{x}_2]}_{\in \mathbb{R}^{2\times 2} \ (\text{or } \mathbb{C}^{2\times 2})} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} =: \Phi(t) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ (def. of Φ)

ii) $\vec{x}_1, \vec{x}_2$ lin. indep. in $\mathbb{R} \Rightarrow \Phi(t)$ invertible for all t

iii) $\vec{x}_0 \overset{(2)}{=} \vec{x}(t_0) = \Phi(t_0)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \Rightarrow \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \Phi^{-1}(t_0)\vec{x}_0$

We have proved:

<u>Lem. 1:</u> The (unique) sol'n of (1) and (2) is

$$\vec{x} = \Phi(t)\,\Phi^{-1}(t_0)\,\vec{x}_0 .$$

Φ is a <u>fundamental-matrix</u> for (1).

<u>Ex. 1:</u>

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

Eig val's / vec's:

$$\lambda_1 = i \;,\quad \vec{v}_1 = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$$

$$\lambda_2 = \overline{\lambda_1} \;,\quad \vec{v}_2 = \overline{\vec{v}_1}$$

Basis:

$$\vec{x}_1 = \operatorname{Re}(\vec{v}_1 e^{\lambda_1 t}) \;,\quad \vec{x}_2 = \operatorname{Im}(\vec{v}_1 e^{\lambda_1 t})$$

Fund. matrices:

$$\Phi_1(t) = [\vec{x}_1(t), \vec{x}_2(t)] \in \mathbb{R}^{2\times 2}$$

$$\Phi_2(t) = [c\vec{x}_1(t), c\vec{x}_2(t)] \in \mathbb{R}^{2\times 2} \quad (c \neq 0)$$

$$\Phi_3(t) = [\vec{v}_1 e^{\lambda_1 t}, \vec{v}_2 e^{\lambda_2 t}] \in \mathbb{C}^{2\times 2}$$

<u>Obs. 2:</u>

i) Many Φ's:

Φ fund. matr. + B inv. $\Rightarrow$ $\Phi \cdot B$ fund. matr.

[HW 2]

ii) $\dot{\Phi} = A\Phi$ [HW 2]

iii) If $\Phi(0) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then

$\Phi(t) = e^{tA}$ (def. of matrix exp.)

[Later – e^{tA} is unique]

B. Stability of solns

Def. 1: $x(t) = \varphi(t; x_0, t_0)$ is the sol'n of (1) and (2).

Obs. 3: $\varphi(t; x_0, t_0) = \Phi(t) \cdot \Phi^{-1}(t_0) \cdot \vec{x}_0$

Called flow of (1).

Def. 2: Stability of sol'ns

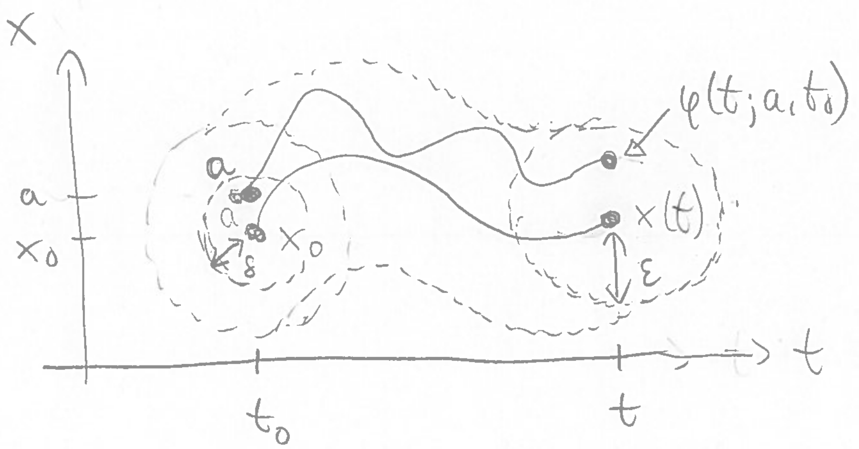
i) $x(t) = \varphi(t; x_0, t_0)$ is stable (forward in time) if for all $\varepsilon > 0$, there is $\delta > 0$ s.t.

(*) $|x_0 - a| < \delta \Rightarrow |x(t) - \varphi(t; a, t_0)| < \varepsilon$ for all $t > t_0$

ii) $x(t) = \varphi(t; x_0, t_0)$ is asymptotically stable if it is stable and there are $\delta > 0$ s.t.

(**) $|x_0 - a| < \delta \Rightarrow \lim_{t \to \infty} |x(t) - \varphi(t; a, t_0)| = 0$

iii) $x(t)$ is unstable if not stable.



Rem. 2:

i) Stable: All sol'ns starting sufficiently near x_0 must remain near $x(t)$

ii) Unstable: Enough that one sol'n do not remain near $x(t)$

iii) There are (non-lin.) ODEs where (**) holds and the sol'ns are unstable!

By Obs. 3:

$$|\varphi(t;a,t_0) - \varphi(t;b,t_0)|$$

$$= |\Phi(t)\Phi^{-1}(t_0)\cdot(a-b)|$$

$$\leq \|\Phi(t)\|\cdot\|\Phi^{-1}(t_0)\|\cdot|a-b| \quad (2\times |Ax| \leq \|A\||x|)$$

So since a, b arbitrary:

Thm. 1:

a) $\|\Phi(t)\| \leq M < \infty$ for all $t > t_0 \Rightarrow$ all sol'ns of (1) are stable

b) $\lim_{t\to\infty} \|\Phi(t)\| = 0 \quad \Rightarrow$ —— " —— asympt. stable

Since $\Phi(t) = [x_1(t), x_2(t)]$ and

$$\|\Phi(t)\| \le 2 \max_{i,j=1,2} |x_{i,j}(t)|,$$

it follows that

$$\|\Phi(t)\| \xrightarrow[t\to\infty]{} \infty \Rightarrow \text{either } |x_1| \to \infty \text{ or } |x_2| \to \infty$$

And if $|x_1(t)| \xrightarrow[t\to\infty]{} \infty$, then

$$|\varphi(t; x_0, t_0) - \underbrace{\varphi(t; x_0 + \delta x_1(t_0), t_0)}_{= \varphi(t; x_0, t_0) + \delta x_1(t)}|$$

(by linearity and uniqueness

$$= \delta |x_1(t)| \xrightarrow[t\to\infty]{} \infty \quad \text{for all } \delta > 0.$$

Hence, since x_0 is arbitrary:

<u>Thm. 2:</u>

$$\lim_{t\to\infty} \|\Phi(t)\| = \infty \Rightarrow \underline{\text{all}} \text{ sol'ns of (7) are unstable}$$

<u>Ex. 3:</u>

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \overset{\text{last time}}{\Longrightarrow} \text{ basis } \vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}, \vec{x}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t}$$

$$|x_2| \xrightarrow[t\to\infty]{} \infty \Rightarrow \underline{\text{all}} \text{ sol'ns unstable}$$

<u>Obs. 4:</u> $\|A\| \ge \max_{i,j=1,2} |a_{ij}|$ (HW 2) so

$$|x_1| \xrightarrow[t\to\infty]{} \infty \Rightarrow \|[x_1, x_2]\| \xrightarrow[t\to\infty]{} \infty$$

Let $\lambda_j / \vec{v}_j$ be eig.val's/vec's of A:

i) $|\vec{v}_j e^{\lambda_j t}| = |\vec{v}_j| \cdot |e^{\lambda_j t}| \underset{\uparrow}{=} |\vec{v}_j| \, e^{Re \lambda_j t}$, $j = 1, 2$

$|e^{i Im \lambda_j t}| = 1$

ii) $|(\vec{u} + t\vec{v}_1) e^{\lambda_1 t}| = |\vec{u} + t \vec{v}_1| \cdot e^{Re \lambda_1 t}$

Since all basis func'ns of (1) are of this form, Thm. 1 and 2 gives:

Thm. 3: Let λ_1, λ_2 be eig. val's of A.

i) All sol'ns of (1) stable $\Rightarrow \max\limits_{j=1,2} Re \lambda_j \leq 0$

ii) $\max\limits_{j=1,2} Re \lambda_j \leq 0$ and $\lambda_1 \neq \lambda_2$ ($\Rightarrow \vec{v}_1, \vec{v}_2$ lin. indep.)

$\Rightarrow$ all sol'ns of (1) stable.

iii) $\max\limits_{j=1,2} Re \lambda_j < 0 \Rightarrow$ all sol'ns of (1) asympt. stab.

iv) $\max\limits_{j=1,2} Re \lambda_j > 0 \Rightarrow$ all sol'ns of (1) unstab.

Ex. 4:

$A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \Rightarrow \lambda = \pm i \Rightarrow Re \lambda_j = 0$

By Thm. 3, all sol'ns of (1) are stable.