Diff. likn. 31.1.13

22 tilstede

~5 min transp. + info

A. General lin. n×n syst.

(1) $\dot{x} = A(t)x + b(t)$

(2) $x(t_0) = x_0$

where $A: \mathbb{R} \to \mathbb{R}^{n\times n}$; $b: \mathbb{R} \to \mathbb{R}^n$; $x: \mathbb{R} \to \mathbb{R}^n$,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Assume (always!): $A(t), b(t)$ cont. for $t \in \mathbb{R}$.

[$\Leftrightarrow$ $a_{ij}(t), b_i(t)$ cont. in $\mathbb{R}$, $i,j = 1,\dots,n$]

Obs.1:

i) Flow $\varphi(t; x_0, t_0)$ = sol'n of (1) and (2)

ii) Inhomogenous ($b \not\equiv 0$), non-autonomous (A, b not const.), normal form ($\dot{x} = \dots$)

Ex. 1: If A not cont. in $\mathbb{R}$

$$\ddot{x} = -\frac{2}{t-1}\dot{x} \iff \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{bmatrix}0 & 1\\ 0 & -\frac{2}{t-1}\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

Special sol'n: $x = \frac{1}{t-1}$, def. only for $t<1$ $[t_0<1]$
or $t>1$ $[t_0>1]$

Sol'n do not exist for all $t \in \mathbb{R}$! (x not cont.)

Tm. 1: (Existence - A, b cont. in $\mathbb{R}$)
There exits a sol'n $x(t)$ of (1) and (2) for all $t \in \mathbb{R}$.

Pf: Later (non-lin. theory)

Lem. 1:

$$|\varphi(t; x_0, t_0) - \varphi(t; x_1, t_0)| \leq e^{K_T t} |x_0 - x_1|, \quad t \in [t_0, T],$$

where $K_T = \max_{s \in [t_0, T]} \|A(s)\| < \infty$ and $T \geq t_0$.

Rem. 1:

i) $\|A\| = \max_{|x|=1} |Ax|$, $\quad |Ax| \leq \|A\| \cdot |x|$

ii) $\max_{i,j} |a_{ij}| \leq \|A\| \leq n \max_{i,j} |a_{ij}|$

Pf.:

Def.: $x(t) = \varphi(t; \underset{\sim}{x_0}, t_0)$, $y(t) = \varphi(t; \underset{\sim}{x_1}, t_0)$,

$$\sigma(t) = |x(t) - y(t)|^2$$

$$\frac{d}{dt}\sigma = 2(x-y)\cdot(\dot{x}-\dot{y}) \overset{(1)}{=} 2(x-y)\cdot A(t)(x-y)$$

C-S + $|Ax| \leq \|A\| \cdot |x|$

$$\leq 2\|A(t)\| \cdot |x-y|^2 \leq 2K_T\sigma, \; t \in [t_0, T]$$

$\Downarrow$ int. factor

$$\frac{d}{dt}\left(e^{-2K_T t}\sigma\right) = e^{-2K_T t}\left(\dot{\sigma} - 2K_T\sigma\right) \leq 0, \; t \in [t_0, T]$$

$\Downarrow$ int. $\int_{t_0}^{t}$

$$e^{-2K_T t}\sigma(t) \leq e^{-2K_T t_0}\sigma(t_0), \quad t \in [t_0, T] \qquad \square$$

Thm. 2: (uniqueness)

There is not more than one sol'n of (1) and (2) for all $t \in \mathbb{R}$.

Pf.: By Lem. 1 as in 2×2 pf. $\square$

B. Homogeneous syst. ($b \equiv 0$)

(3) $\quad \dot{x} = A(t)x \quad ; \quad A: \mathbb{R} \to \mathbb{R}^{n\times n}$, $x: \mathbb{R} \to \mathbb{R}^n$

Rem. 1:

i) $x_j(t)$ solve (3), $c_j \in \mathbb{R}$, $j = 1, \dots, k$

$\Rightarrow \sum c_j x_j(t)$ solves (3) (linearity)

lin. dep.
lin. indep.

ii) Basis: n lin. indep. sol'ns of (3); $x_1, \dots, x_n$.

Gen. sol'n: $c_1 x_1 + c_2 x_2 + \dots + c_n x_n$; $c_1, \dots, c_n \in \mathbb{R}$

$x_1, \dots, x_k$ lin. indep. func'ns if not lin. dep.

$x_1, \dots, x_k$ lin. indep. func'ns in $I \subset \mathbb{R}$ if there is $c = (c_1, \dots, c_k) \in \mathbb{R}^k$, $c \neq 0$, s.t. $\sum_{i=1}^{k} c_i x_i = 0$ for all $t \in I$.

iii) Fund. matrix: $\Phi = [x_1, \dots, x_n] : \mathbb{R} \to \mathbb{R}^{n \times n}$ (collumns)

when $x_1, \dots, x_n$ basis for (3).

iv) $\dot{\Phi} = [\dot{x}_1, \dots, \dot{x}_n] \overset{(3)}{=} [Ax_1, \dots, Ax_n] = A\Phi$

and Φ invertible ($x_1, \dots, x_n$ lin. indep.) [Lem. 2b]

v) Φ fund. matr. + B inv. $\Rightarrow \Phi \cdot B$ fund. matr.

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(4) $\dot{\Phi} = A\Phi$; $A, \Phi : \mathbb{R} \to \mathbb{R}^{n \times n}$

(5) $\Phi(t_0) = \Phi_0$

Def. 1: $\Phi : \mathbb{R} \to \mathbb{R}^{n \times n}$ fund. matr. for (3) if invertible sol'n of (4) for all $t \in \mathbb{R}$.

Lem. 2:

a) There exists a unique sol'n Φ of (4) and (5).

b) Φ_0 invertible $\Rightarrow$ Sol'n $\Phi(t)$ of (4) and (5) invertible for all $t \in \mathbb{R}$

c) $\Phi = [x_1, \dots, x_n]$ fund. matrix for (3) $\Leftrightarrow$ $x_1, \dots, x_n$ sol'n basis for (3)

Pf.:

columns

$\Phi = [x_1, \dots, x_n]$

1.) (4) + (5) $\Leftrightarrow$ $\dot{x}_i = A x_i$, $x_i(t_0) = \Phi_{0,i}$, $i = 1, \dots, n$

Hence Tm. 1 and 2 $\Rightarrow$ a)

2.) If $\Psi = \Phi^{-1}$ exists for all $t \in \mathbb{R}$, then

$$\Psi \cdot \Phi = I \quad \text{in } \mathbb{R}$$

$$\Rightarrow 0 = \frac{d}{dt}(\Psi \cdot \Phi) = \dot{\Psi} \cdot \Phi + \Psi \cdot \dot{\Phi} \overset{(4)}{=} \dot{\Psi} \cdot \Phi + \Psi A \Phi \; \Big| \cdot \Psi$$

(*) $\Rightarrow \dot{\Psi} = -\Psi A$ or $\dot{\Psi}^T = -A^T \Psi^T$

3.) By a), there is unique sol'n Ψ of (*) and

$$\Psi(t_0) = \Phi_0^{-1}$$

Chk.: $\frac{d}{dt}(\Psi \cdot \Phi) = \dot{\Psi} \cdot \Phi + \Psi \cdot \dot{\Phi} = -\Psi A \Phi + \Psi A \Phi = 0$

$$(\Psi \cdot \Phi)(t_0) = \Phi_0^{-1} \cdot \Phi_0 = I$$

$\Rightarrow \Psi \cdot \Phi(t) = I$ for all $t \in \mathbb{R}$ $\Rightarrow$ b) $\square$

4.) Φ fund. matrix $\Rightarrow$ $\dot{x}_i = A x_i$ (Rem. 1 iv) and $x_1, \dots, x_n$ lin. indep. (since lin. indep. pt. wise (invertible))

$\Rightarrow x_1, \dots, x_n$ basis.

$x_1, \dots, x_n$ basis $\Rightarrow$ Φ solve (4) (Rem. 1 iv)), inv.

($x_1, \dots, x_n$ lin. indep

$\Rightarrow$ lin. indep. at one pt.

$\Rightarrow$ lin. indep. and inv. everywhere

by b))

Hence c) holds $\square$

If $\Phi_0 = I$, then by Lem. 2:

Thm. 3:

a) There (always) exists a fund. matr. for (3).

b) ~~There~~ " ———— basis for (3).

Ex. 2:

$$\dot{x} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 2te^{-t} & 0 & 1 \end{bmatrix} x, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\left.\begin{aligned} \Rightarrow x_1 &= C_1 e^{2t} \\ x_2 &= C_2 e^{-t} \end{aligned}\right\} \text{ uncoupled eq'ns}$$

$$\dot{x}_3 = 2te^{-t} x_1 + x_3$$

$$\Rightarrow \dot{x}_3 - x_3 = 2t \cdot e^{-t} \cdot C_1 e^{2t} \quad \Big| \cdot e^{-t}$$

int. factor

$$\Rightarrow \frac{d}{dt}(e^{-t} x_3) = 2C_1 t$$

int.

$$\Rightarrow e^{-t} x_3 = C_1 t^2 + C_3$$

Gen. sol'n

$$x = C_1 \begin{bmatrix} e^{2t} \\ 0 \\ t^2 e^t \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ e^{-t} \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}$$

Fund. matrix:

$$\Phi = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ t^2 e^t & 0 & e^t \end{bmatrix} \qquad (\Phi(0) = I \text{, inv.!})$$

<u>Obs. 2:</u>

$\Phi = [x_1, \dots, x_n]$

i) $x(t) := \Phi(t) \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = c_1 x_1 + \dots + c_n x_n$ solves (3) [Rem. 1 iv)]

ii) $x(t_0) = \Phi(t_0) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = x_0 \Rightarrow \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \Phi^{-1}(t_0) x_0$

so

<u>Thm. 4:</u>

a) If Φ is a fund. sol'n for (3), then the unique sol'n x of (2) and (3) is given by

$$x(t) = \Phi(t) \cdot \Phi(t_0)^{-1} \cdot x_0$$

b) If $x_1, \dots, x_n$ sol'n basis for (3), then for every sol'n x of (3) there is $(c_1, \dots, c_n) \neq 0$ s.t.

$$x = c_1 x_1 + \dots + c_1 x_n \quad \left(= \Phi \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \right)$$

[b): by a) we can take $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \Phi^{-1}(0) x(0)$]

<u>Ex. 3:</u>

Solve (*) of Ex. 2 and $x(0) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$\text{Ex. 2} \Rightarrow \Phi(t) = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ t^2 e^t & 0 & e^t \end{bmatrix}$$

$$x(t) = \Phi(t) \underbrace{\Phi(0)^{-1}}_{= I} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \overset{\text{chk.}}{=} \underline{\begin{bmatrix} 0 \\ e^{-t} \\ -e^t \end{bmatrix}}$$