1a) $A=\begin{pmatrix}0 & 2\\ -1 & 3\end{pmatrix}$ EGENVERDIENE: $\det\begin{pmatrix}-\lambda & 2\\ -1 & 3-\lambda\end{pmatrix} = (-\lambda)(3-\lambda)+2$

$$= \lambda^2-3\lambda+2$$

$$\Rightarrow \det(A-\lambda I)=0 \Leftrightarrow \lambda^2-3\lambda+2=0$$

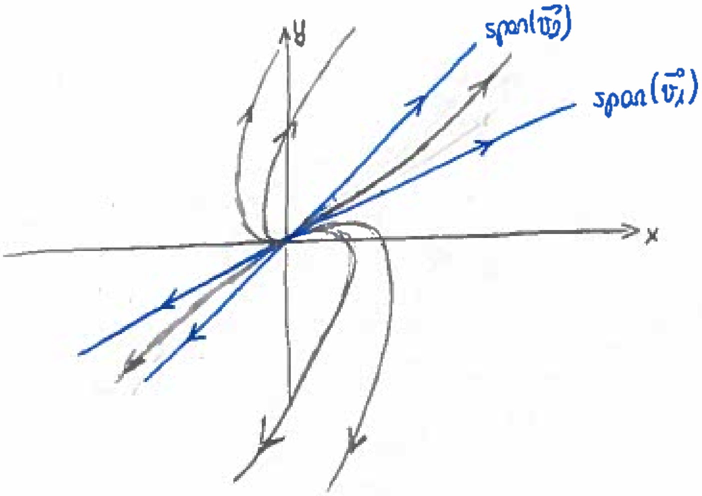
$$\lambda_{1,2} = \frac{3}{2}\pm\sqrt{\frac{9}{4}-2} = \frac{3}{2}\pm\frac{1}{2} = \begin{matrix}1\\2\end{matrix} \Rightarrow \text{USTABIL NODE}$$

EGENVEKTORER: $\lambda_1=1$ $\begin{pmatrix}-1 & 2\\ -1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\vec{0}$

$\Rightarrow$ EGENVEKTOR $\vec{v_1}=\begin{pmatrix}2\\1\end{pmatrix}$

$\lambda_2=2$ $\begin{pmatrix}-2 & 2\\ -1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\vec{0}$

$\Rightarrow$ EGENVEKTOR $\vec{v_2}=\begin{pmatrix}1\\1\end{pmatrix}$



1b) LA $P=(\vec{v_1},\vec{v_2})=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} \Rightarrow A=P\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}P^{-1} = PDP^{-1}$

$$\Rightarrow \begin{tikzcd} \dot{\vec{x}}=A\vec{x} \arrow[r, bend left, "\vec{z}=P^{-1}\vec{x}"] & \dot{\vec{z}}=D\vec{z} \arrow[l, bend left, "\vec{x}=P\vec{z}"] \end{tikzcd}$$

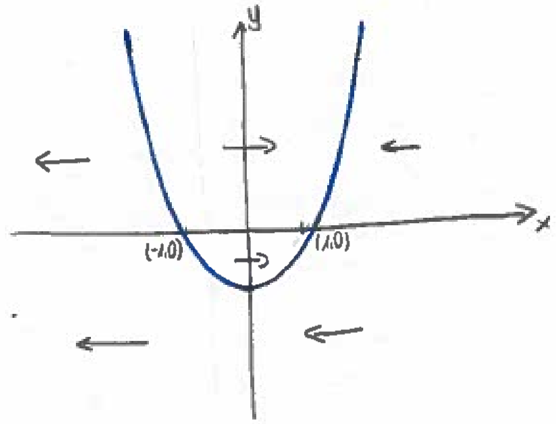
$\dot{\vec{z}}=D\vec{z}$ HAR DEN GENERELLE LØSNINGEN $\vec{z}(t)=\begin{pmatrix}c_1e^{t}\\ c_2e^{2t}\end{pmatrix}$ $(c_1,c_2\in\mathbb{R})$

$\Rightarrow$ $\dot{\vec{x}}=A\vec{x}$ HAR DEN GENERELLE LØSNINGEN:

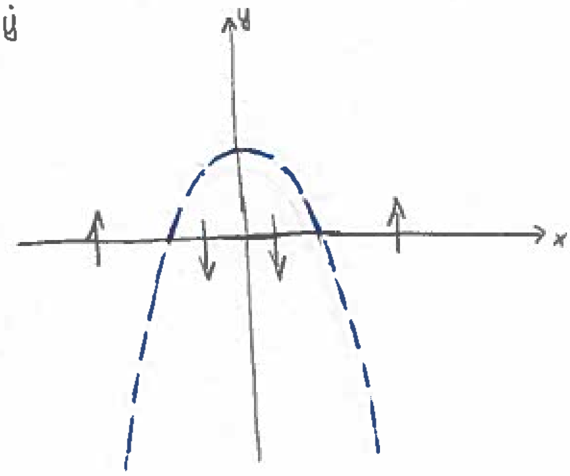
$$\vec{x}(t)=P\vec{z}(t)=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}c_1e^{t}\\ c_2e^{2t}\end{pmatrix} = c_1e^{t}\begin{pmatrix}2\\1\end{pmatrix}+c_2e^{2t}\begin{pmatrix}1\\1\end{pmatrix} \qquad (c_1,c_2\in\mathbb{R})$$

2a) $\dot{x} = y - x^4 + 1$ $\qquad\qquad$ $\dot{y} = y + x^4 - 1$

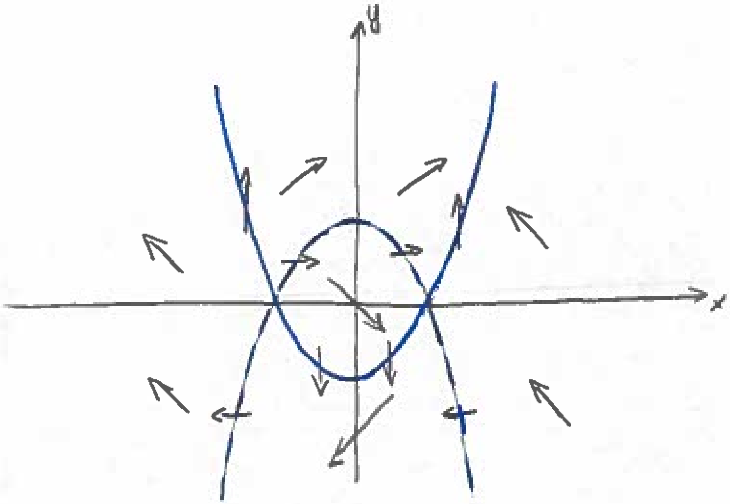
$\dot{x}$ $\qquad\qquad$ $\dot{y}$



$(\text{—} \dots \dot{x} = 0)$



$(\text{- - -} \dots \dot{y} = 0)$



$\Rightarrow$ TO LIKEVEKTSPUNKTER:

$(-1,0)$

$(1,0)$.

(1,0)

$$\dot{x} = y-(x^4-1) = y-(x-1)(1+x+x^2+x^3) = y-(x-1)(4+6(x-1)+4(x-1)^2+(x-1)^3) = y-4(x-1)+O((x-1)^2)$$

$$\dot{y} = y+x^4-1 = y+(x-1)(1+x+x^2+x^3) = y+(x-1)(4+6(x-1)+4(x-1)^2+(x-1)^3) = y+4(x-1)+O((x-1)^2)$$

$$\Rightarrow \begin{pmatrix} x-1 \\ y \end{pmatrix}^{\circ} = \underbrace{\begin{pmatrix} -4 & 1 \\ 4 & 1 \end{pmatrix}}_{\text{MATRISE TIL LINEARISERINGEN}} \begin{pmatrix} x-1 \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} O((x-1)^2) \\ O((x-1)^2) \end{pmatrix}}_{O((x-1)^2+y^2)} \rightarrow (1,0) \text{ SADEL.}$$

$\Downarrow$

EGENVERDIER:

$$\lambda_{1,2} = -\frac{3}{2} \pm \sqrt{\frac{9}{4}+8}$$

$\downarrow$

SADEL

$(-1,0)$:

$$\dot{x} = y-(x^4-1) = y-(x^2-1)(x^2+1) = y-(x+1)((x+1)-2)((x+1)^2-2(x+1)+2) = y+4(x+1)+\mathcal{O}((x+1)^2)$$
$$\dot{y} = y+(x^4-1) = y+(x^2-1)(x^2+1) = y+(x+1)((x+1)-2)((x+1)^2-2(x+1)+2) = y-4(x+1)+\mathcal{O}((x+1)^2)$$

$$\Rightarrow \begin{pmatrix} x+1 \\ y \end{pmatrix}^{\circ} = \underbrace{\begin{pmatrix} 4 & 1 \\ -4 & 1 \end{pmatrix}}_{\text{MATRISE TIL LINEARISERINGEN}} \begin{pmatrix} x+1 \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} \mathcal{O}((x+1)^2) \\ \mathcal{O}((x+1)^2) \end{pmatrix}}_{\mathcal{O}((x+1)^2+y^2)} \quad \Rightarrow (-1,0) \text{.. USTABIL SPIRAL}$$

$\Downarrow$

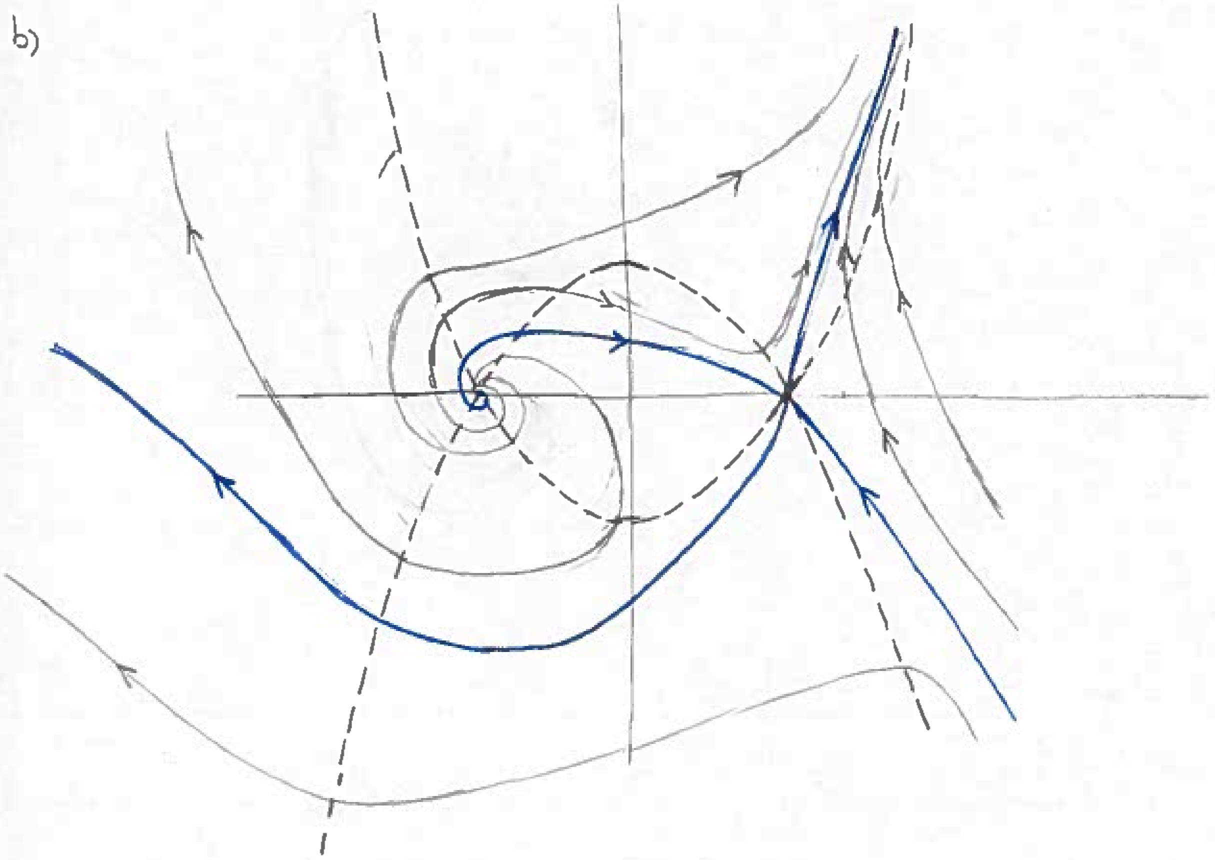
EGENVERDIER

$$\lambda_{1,2} = \frac{5}{2} \pm \sqrt{\frac{25}{4}-8}$$

$\Downarrow$

USTABIL SPIRAL

b)



3a) $\dot{x} = 5x + y - 4z$
$\dot{y} = 2y$
$\dot{z} = 4x + y + 5z$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}^{\bullet} = \begin{pmatrix} 5 & 1 & -4 \\ 0 & 2 & 0 \\ 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

EGENVERDIER: $(5-\lambda)(5-\lambda)(2-\lambda) - (2-\lambda)(-4)4$
$= (2-\lambda)(25 - 10\lambda + \lambda^2 + 16)$
$= (2-\lambda)(\underbrace{\lambda^2 - 10\lambda + 41})$

NP $\lambda_{1,2} = 5 \pm \sqrt{25-41} = 5 \pm 4i$

$\Rightarrow$ EGENVERDIER: $\lambda_1 = 2$, $\lambda_2 = 5+4i$, $\lambda_3 = 5-4i$ $\Rightarrow$ $(0,0)$ USTABIL

b) $\dot{x} = \frac{1-3t^2}{1+t^2} x + (e^{-t} + 2) y = -3x + \frac{4}{1+t^2} x + (e^{-t} + 2) y$
$\dot{y} = \frac{1}{1+t^4} x - 4y$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}^{\bullet} = \underbrace{\begin{pmatrix} -3 & +2 \\ 0 & -4 \end{pmatrix}}_{=A} \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{4}{1+t^2} & e^{-t} \\ \frac{1}{1+t^4} & 0 \end{pmatrix}}_{C(t)} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow (0,0) \text{ ASYMPTOTISK STABIL}$$

MED EGENVERDIENE
-3 & -4
$\Downarrow$
ASYMPTOTISK STABIL

$$\int_0^\infty \frac{4}{1+t^2} + \frac{1}{1+t^4} + e^{-t}\, dt < \infty$$

$$\left[\int_0^\infty \frac{4}{1+t^2} dt \le \int_0^1 4\, dt + \int_1^\infty \frac{4}{t^2} dt = 4 + 4 = 8 \right.$$
$$\int_0^\infty \frac{1}{1+t^4} dt \le \int_0^1 1\, dt + \int_1^\infty \frac{1}{t^2} dt = 2$$
$$\left. \int_0^\infty e^{-t} dt = 1 \right]$$

c) $\dot{x} = 3y - x^3 - xy^2$
$\dot{y} = -3x - y^3 + x^2 y$

$\Rightarrow$ $\left.\begin{aligned} \dot{x}x &= 3xy - x^4 - x^2y^2 \\ \dot{y}y &= -3xy - y^4 + x^2y^2 \end{aligned}\right\} \Rightarrow \dot{x}x + y\dot{y} = -x^4 - y^4$

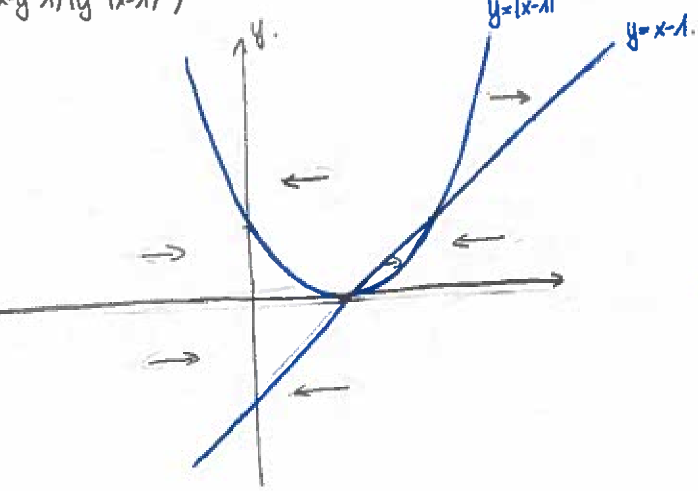
$\Rightarrow$ LA $V(x,y) = \frac{1}{2}(x^2+y^2)$ $\Rightarrow$ $V(0,0) = 0$, $\dot{V}(x,y) = -x^4 - y^4 < 0$ $\forall (x,y) \neq (0,0)$

$\Rightarrow$ V STERK LYAPUNOV FKT

$\Rightarrow$ $(0,0)$ ASYMPTOTISK STABIL

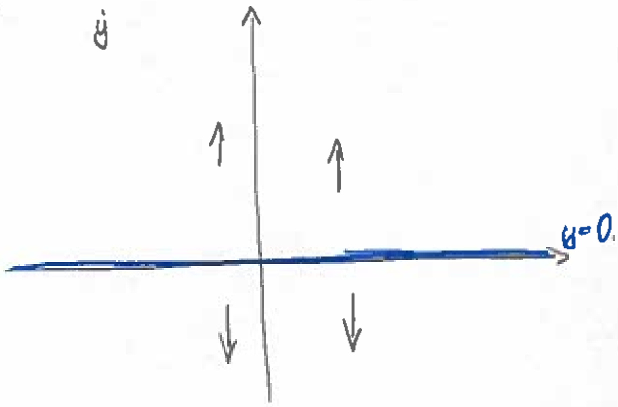
4) $\dot{x} = (x-y-1)(y-(x-1)^2)$

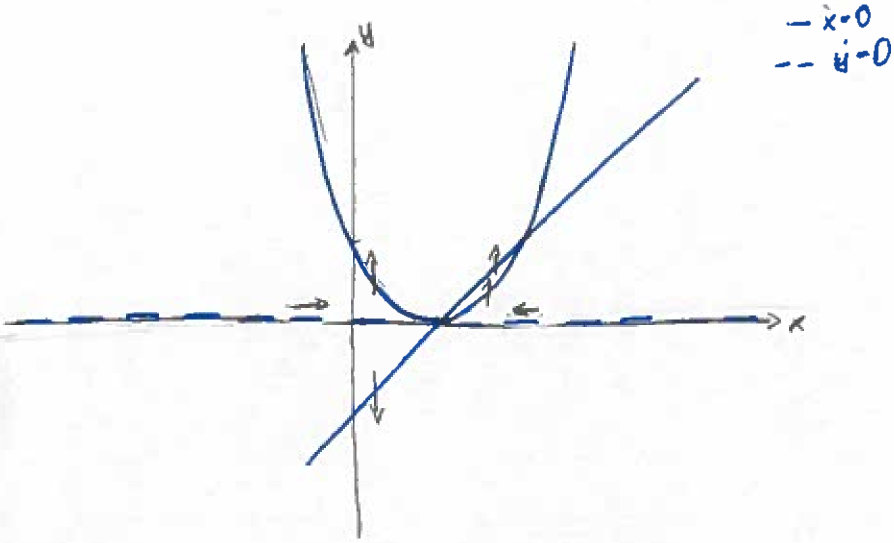
$\dot{x}$



$\dot{y} = y$

$\dot{y}$





$\Rightarrow$ INDEX $= -1$ $(\leftarrow \nearrow \uparrow \nearrow \uparrow \nearrow \uparrow \nearrow \rightarrow \searrow \downarrow \searrow \leftarrow)$

5) $\dot{x} = x^3 - x^2y + 2x^3y = f(x,y) \Rightarrow f_x(x,y) = 3x^2 - 2xy + 6x^2y$

$\dot{y} = xy^2 - 3x^2y^2 + y = g(x,y) \Rightarrow g_y(x,y) = 2xy - 6x^2y + 1$

$\Big\} \Rightarrow (f_x + g_y)(x,y) = 1 + 3x^2 > 0 \quad \forall (x,y) \in \mathbb{R}^2$

$+$ $\mathbb{R}^2$ SAMMENHENGENDE

INGEN IKKE-KONSTANTE PERIODISKE LØSNINGER P.G.A. BENDIXSONS NEGATIV KRITERIUM.

6.) EN LØSNING $x(t)$ ER ASYMPTOTISK STABIL HVIS DET FINNES $\varepsilon>0$ SÅ

$$\|x(0)-y(0)\|<\varepsilon \Rightarrow \lim_{t\to\infty}\|x(t)-y(t)\|=0$$

LA $V(x,y)=\lambda x^2+\mu y^2 \Rightarrow \dot V(x,y)=2\lambda x\dot x+2\mu y\dot y$

$$=2\lambda^2x^2+2\mu^2y^2+2\lambda x f(x,y)+2\mu y g(x,y)$$

·) $\lim_{x^2+y^2\to 0}\frac{f(x,y)^2+g(x,y)^2}{x^2+y^2}=0 \Rightarrow \forall\varepsilon>0\ \exists\delta>0$ SÅ

$$\frac{f(x,y)^2+g(x,y)^2}{x^2+y^2}<\varepsilon \quad \forall \|(x,y)\|<\delta$$

$$\Rightarrow \dot V(x,y)\geq 2\lambda^2x^2+2\mu^2y^2-\lambda^2x^2-f(x,y)^2-\mu^2y^2-g(x,y)^2$$

$$\geq \lambda^2x^2+\mu^2y-f(x,y)^2-g(x,y)^2$$

$$\geq \lambda^2(x^2+y^2)-\varepsilon(x^2+y^2)$$

$$=(\lambda^2-\varepsilon)(x^2+y^2)>0 \quad \text{IF WE CHOOSE } \varepsilon=\tfrac{\lambda^2}{2}!$$

·) $f(x,y)=-2\mu x^3+O(x^2+y^2)$ NÅR $x^2+y^2\to\infty$.

$\Rightarrow \exists M>0$ OG $\hat R>0$ SLIK AT

$$|f(x,y)+2\mu x^3|\leq M|x^2+y^2| \quad \forall \|(x,y)\|>\hat R$$

$g(x,y)=-2\mu y^3+O(x^2+y^2)$ NÅR $x^2+y^2\to\infty$

$\Rightarrow \exists N>0$ OG $\tilde R>0$ SLIK AT

$$|g(x,y)+2\mu y^3|\leq N|x^2+y^2| \quad \forall \|(x,y)\|>\tilde R$$

LA $R=\max(\tilde R,\hat R) \Rightarrow |f(x,y)+2\mu x^3|\leq M|x^2+y^2|$ OG $|g(x,y)+2\mu y^3|\leq N|x^2+y^2| \quad \forall \|(x,y)\|>R$

$$\Rightarrow \dot V(x,y)\leq 2\lambda^2x^2+2\mu^2y^2+2\lambda x(-2\mu x^3+f(x,y)+2\mu x^3)+2\mu y(-2\mu y^3+g(x,y)+2\mu y^3)$$

$$\leq 2\lambda^2 x+2\mu^2y^2-2\lambda\mu x^4-2\mu^2y^4+2|\lambda||x|M|x^2+y^2|+2\mu|y|N|x^2+y^2|$$

$$\leq 2\mu^2(x^2+y^2)-2\lambda^2(x^4+y^4)+Mx^2+\lambda^2|x^2+y^2|+Ny^2+\mu^2|x^2+y^2|$$

$$\leq (4\mu^2+M+N)(x^2+y^2)-2\lambda^2(x^4+y^4)\overset{?}{<}0$$

MÅ FINNE $C>R$ SLIK AT $(4\mu^2+M+N)(x^2+y^2)<2\lambda^2(x^4+y^4) \quad \forall\ (x^2+y^2)>C$

$$\Rightarrow C(x^2+y^2)<(x^2+y^2)^2=(x^4+2x^2y^2+y^4)\leq 2(x^4+y^4)$$

$\Rightarrow$ HVIS $\frac{4\mu^2+M+N}{\lambda^2}<C \Rightarrow (4\mu^2+M+N)(x^2+y^2)<2\lambda^2(x^4+y^4)$

VELG $\Omega=\{(x,y)\ \ \frac{\lambda^2}{2}<\|(x,y)\|<\sqrt{C}\} \Rightarrow$ INGEN LIKEVEKTSPUNKTER I Ω

$\Rightarrow$ POINCARE-BENDIXSON: $\exists$ IKKE-KONSTANT PERIODISK LØSNING I Ω.