**Problem 1** Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Sketch the phase diagram of the system with orientations.

**Problem 2** Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda^2 & -\lambda^2 - \frac{1}{4} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Find all bifurcation points of the above system.

**Problem 3** Consider the system

$$\begin{aligned} \dot{x} &= y + 4 \\ \dot{y} &= y + x^2 + 2x - 4. \end{aligned}$$

**a)** Find and classify all equilibrium points of the system.

**b)** Sketch the phase diagram of the system with orientations.

**Problem 4** Consider the system

$$\begin{aligned} \dot{x} &= (y - x)(y + x^2) \\ \dot{y} &= (y + x)(y - x^2) \end{aligned}$$

**a)** Compute the index of the origin.

**b)** Determine whether or not the system has closed phase paths surrounding the origin but no other equilibrium points.

**Problem 5** Consider the initial value problem

$$\dot{x} = t^3 x^2, \quad x(t_0) = x_0 \in \mathbb{R}.$$

Find the general solution.
Determine for which pairs $(t_0, x_0)$ there exists a global solution.
Determine for any pair $(t_0, x_0)$ such that $x_0 > 0$ the maximal interval of existence.

**Problem 6** Consider the differential equation

$$\dot{x} = \lambda x + b(t,x), \quad x(0) = x_0 \in \mathbb{R}, \quad \lambda \in \mathbb{R}. \tag{1}$$

Assume that there exists $\gamma(t)$ such that

$$|b(t,x)| \leq \gamma(t)|x|, \quad x, t \in \mathbb{R}.$$

**a)** Show that

$$|x(t)| \leq |x_0| e^{\lambda t + \int_0^t \gamma(s)ds} \quad \text{for all } t \geq 0.$$

**b)** Define what it means for a solution to be asymptotically stable.
Find conditions on $\lambda$ and $\gamma(t)$ which guarantee that all solutions to (1) are asymptotically stable.

**Problem 7** Consider the dynamical system

$$\dot{\vec{x}} = A\vec{x},$$

where $A$ is a $2 \times 2$ matrix, which can be written as $A = PDP^{-1}$, where $P$ is invertible and

$$D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \lambda < \mu < 0.$$

Show with the help of a strong Liapunov function that

$$\lim_{t \to \infty} \|\vec{x}(t)\| = 0$$

for all solutions $\vec{x}(t)$.