1) $\dot{\vec{x}} = A\vec{x}$ WITH $A = \begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix}$

·) EIGENVALUES OF A:

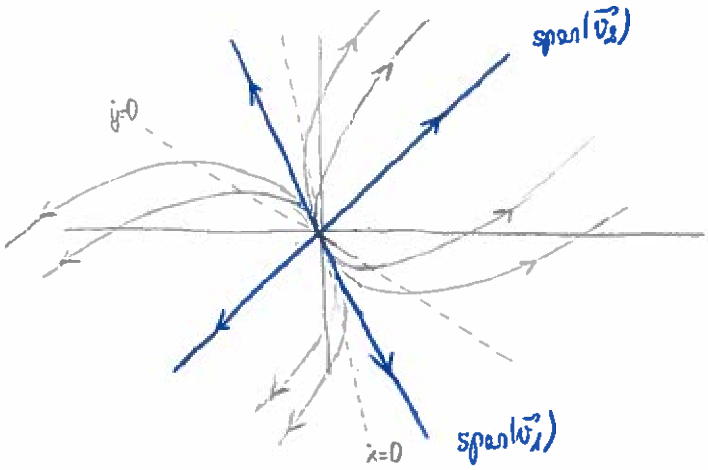
$$(5-\lambda)(3-\lambda)-3=0$$
$$15-3\lambda-5\lambda+\lambda^2-3=0$$
$$\lambda^2-8\lambda+12=0$$
$$\lambda_{1,2} = 4 \pm \sqrt{16-12} = 4 \pm 2$$
$$\lambda_1 = 2,\ \lambda_2 = 6$$

$\rightarrow$ UNSTABLE NODE

·) EIGENVECTORS

$\vec{v}_1$: $\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\vec{v}_2$: $\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \vec{v}_2 = 0 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



2) $\dot{\vec{x}} = A(\lambda)\vec{x}$ WITH $A(\lambda) = \begin{pmatrix} \lambda^2 & -\lambda^2-\frac{1}{4} \\ 1 & -1 \end{pmatrix}$

EIGENVALUES μ_1, μ_2 OF $A(\lambda)$:

$$(\lambda^2-\mu)(-1-\mu)+\lambda^2+\frac{1}{4}=0$$
$$-\lambda^2+\mu-\mu\lambda^2+\mu^2+\lambda^2+\frac{1}{4}=0$$
$$\mu^2+(1-\lambda^2)\mu+\frac{1}{4}=0$$
$$\mu_{1,2} = \frac{\lambda^2-1}{2} \pm \sqrt{\frac{(\lambda^2-1)^2}{4}-\frac{1}{4}} = \frac{\lambda^2-1}{2} \pm \sqrt{\frac{\lambda^2(\lambda^2-2)}{4}}$$

$\Rightarrow$ $\lambda^2-1>0$ IF $|\lambda|>1$ $\qquad$ $\lambda^2(\lambda^2-2)>0$ IF $|\lambda|>\sqrt{2}>1$

$\lambda^2-1<0$ IF $|\lambda|<1$ $\qquad$ $\lambda^2(\lambda^2-2)<0$ IF $|\lambda|<\sqrt{2}$

$\rightarrow$ STABLE SPIRAL IF $|\lambda|<1$

UNSTABLE SPIRAL IF $1<|\lambda|<\sqrt{2}$

UNSTABLE NODE IF $\sqrt{2}<|\lambda|$ (SINCE $0 < \frac{(\lambda^2-1)^2}{4}-\frac{1}{4} < \frac{(\lambda^2-1)^2}{4}$)

$\Rightarrow$ TWO BIFURCATION POINTS $\lambda = \pm 1$.

5a) $\dot{x} = y+4$
$\dot{y} = y + x^2 + 2x - 4$ $\quad (*)$

EP $\left.\begin{array}{l} \dot{x}=0 \Rightarrow y=-4 \\ \dot{y}=0 \Rightarrow y+x^2+2x-4=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y=-4 \\ x^2+2x-8=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y=-4 \\ x=-1\pm\sqrt{1+8}=-1\pm 3 \end{array}\right\} \Rightarrow \begin{array}{l} y=-4 \\ x=-4,2 \end{array}$

TWO EP. $(-4,-4)$, $(2,-4)$

RHS OF $(*)$ CONSISTS OF POLYNOMIALS

$\Rightarrow$ TAYLOR EXPANSION POSSIBLE AROUND EACH POINT IN $\mathbb{R}^2$

$\Rightarrow$ HARTMAN - GROBMAN APPLIES.

$J_{(x,y)} = \begin{pmatrix} 0 & 1 \\ 2x+2 & 1 \end{pmatrix}$

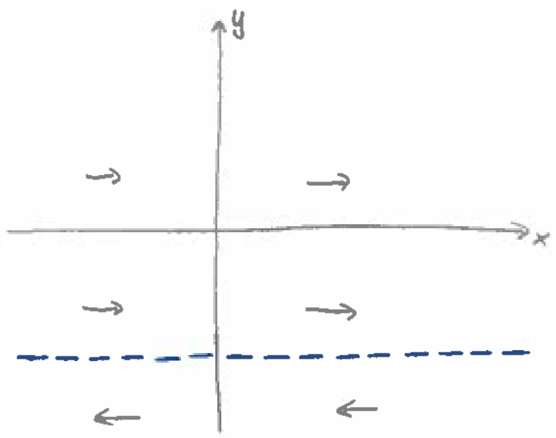
$(-4,-4)$: $J_{(-4,-4)} = \begin{pmatrix} 0 & 1 \\ -6 & 1 \end{pmatrix}$

EIGENVALUES:
$(1-\lambda)(-\lambda)+6=0$
$\lambda^2-\lambda+6=0$
$\lambda_{1,2}=\frac{1}{2}\pm\sqrt{\frac{1}{4}-6}$
$(-4,-4)$: UNSTABLE SPIRAL.

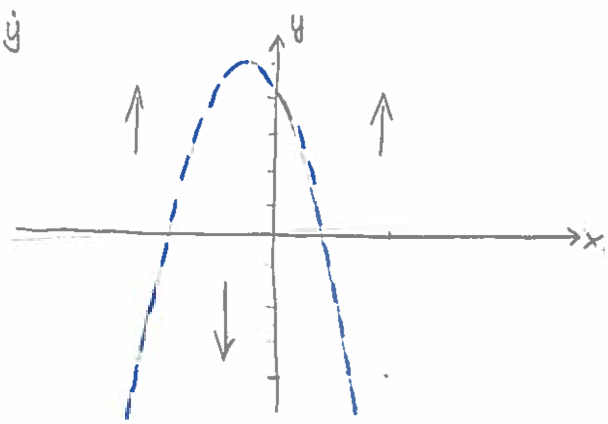
$(2,-4)$: $J_{(2,-4)} = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$

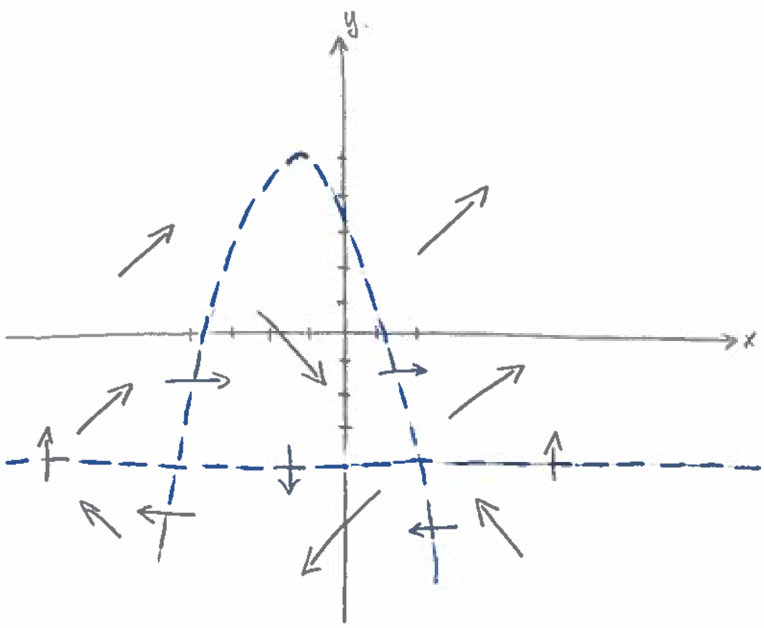
EIGENVALUES:
$(1-\lambda)(-\lambda)-6=0$
$\lambda^2-\lambda-6=0$
$\lambda_{1,2}=\frac{1}{2}\pm\sqrt{\frac{1}{4}+6}$
$(2,-4)$: SADDLE.

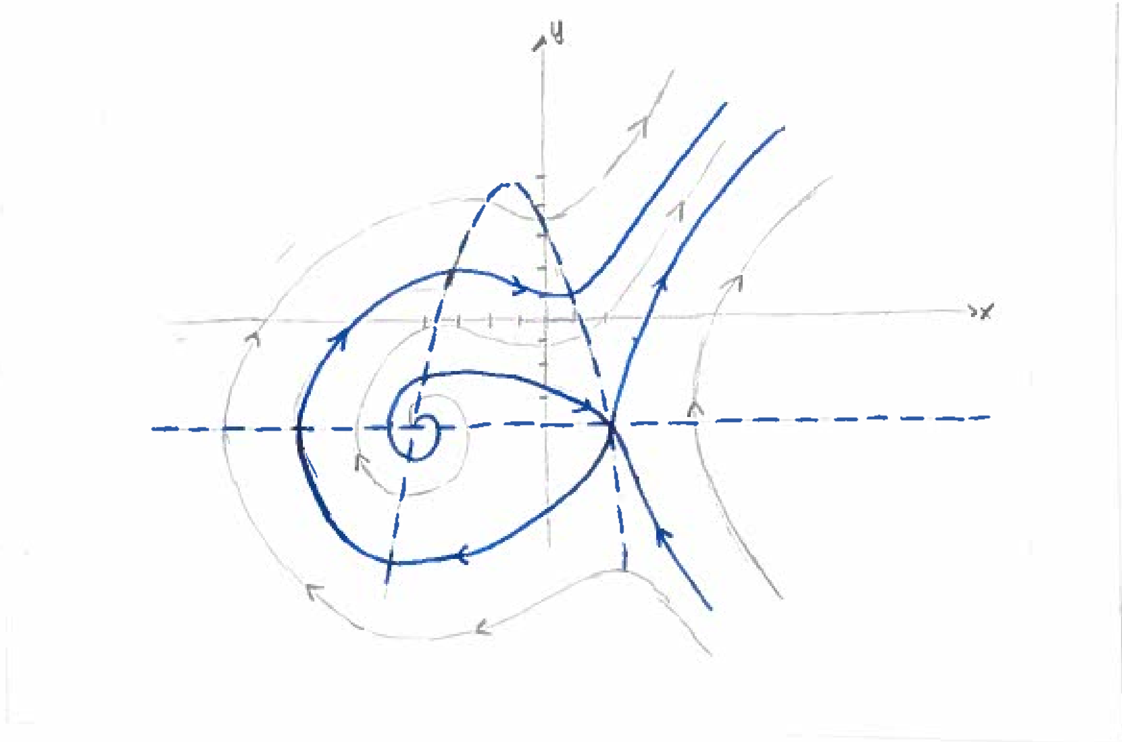
b) $\dot{x}$



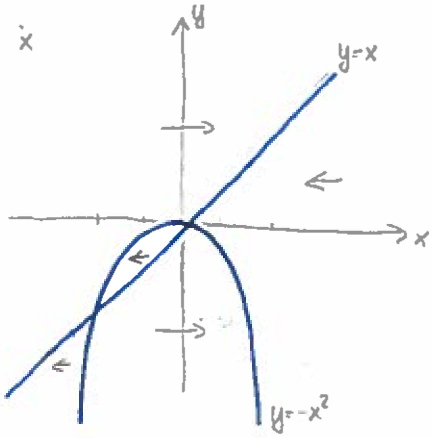
$\dot{y}$

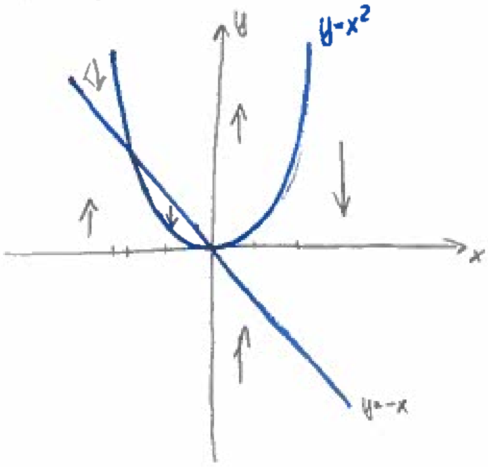


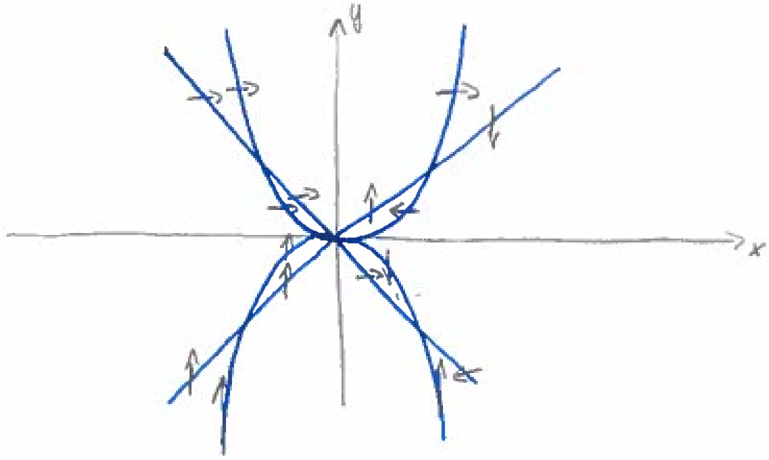




4.) a.) $\dot{x} = (y-x)(y+x^2)$ $\dot{y} = (y+x)(y-x^2)$

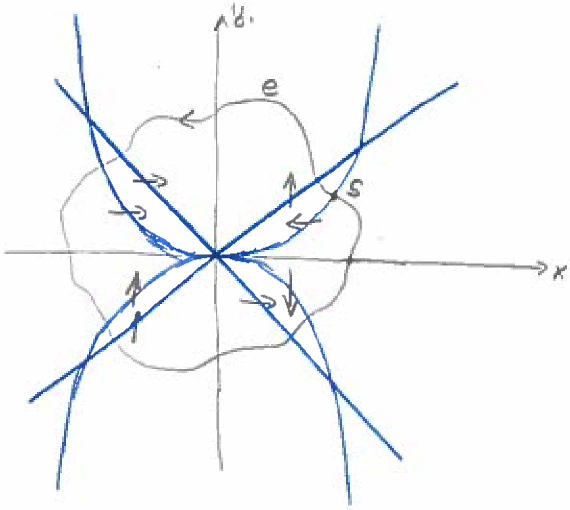






3 EP (0,0), (1,1), (1,-1)

HAVE TO CHOOSE A COUNTER CLOCKWISE CURVE WHICH ONLY SURROUNDS (0,0) BUT NO OTHER EQUILIBRIUM POINT



STARTING AT S WE HAVE:

$\leftarrow \rightsquigarrow \uparrow \rightsquigarrow \rightarrow \rightsquigarrow \rightarrow \rightsquigarrow \uparrow \rightsquigarrow \uparrow \rightsquigarrow \rightarrow \rightsquigarrow \downarrow \rightsquigarrow \leftarrow$

$\rightarrow I_{(0,0)} = -1.$

b) NO, SINCE ANY SIMPLY CLOSED CURVE SURROUNDING (0,0) BUT NO OTHER EP HAS INDEX -1, WHILE ANY CLOSED PHASE PATH ($\triangleq$ PERIODIC SOLUTIONS) HAS INDEX 1.

5.) $\dot{x} = t^3 x^2 \Rightarrow \frac{\dot{x}}{x^2} = t^3 \Rightarrow -\frac{1}{x(t)} + \frac{1}{x(t_0)} = \frac{t^4}{4} - \frac{t_0^4}{4}$

$$4(x(t) - x(t_0)) = (t^4 - t_0^4)\, x(t)\, x(t_0)$$

$$x(t)\left(4 - (t^4 - t_0^4)x(t_0)\right) = 4x(t_0)$$

$$x(t) = \frac{4x(t_0)}{4 - (t^4 - t_0^4)x(t_0)}$$

GLOBAL SOLUTION $\Leftrightarrow 4 - (t^4 - t_0^4)x(t_0) \neq 0 \quad \forall t$

$$4 - (t^4 - t_0^4)x(t_0) = 0 \Leftrightarrow 4 = (t^4 - t_0^4)x(t_0) \Leftrightarrow t^4 = \frac{4 + t_0^4 x(t_0)}{x(t_0)}$$

•) IF $x(t_0) = x_0 > 0$ $\rightsquigarrow$ MAXIMAL INTERVAL OF EXISTENCE

$$\left(-\left(\frac{4 + t_0^4 x(t_0)}{x(t_0)}\right)^{1/4}, \left(\frac{4 + t_0^4 x(t_0)}{x(t_0)}\right)^{1/4}\right)$$

•) IF $x(t_0) = 0 \rightarrow x(t) = 0 \quad \forall t$

•) IF $x(t_0) = x_0 < 0$: GLOBAL SOLUTION IF $4 + t_0^4 x(t_0) > 0$
LOCAL SOLUTION IF $4 + t_0^4 x(t_0) < 0$

$\Rightarrow$ GLOBAL SOLUTION IF $x_0 < 0$ AND $4 + t_0^4 x(t_0) > 0$.

(IF $x(t_0) < 0$ AND $4 + t_0^4 x(t_0) = 0 \Rightarrow \frac{1}{x(t_0)} = -\frac{t_0^4}{4} \Rightarrow -\frac{1}{x(t)} = \frac{t^4}{4} \Rightarrow x(t) = -\frac{4}{t^2} \Rightarrow$ LOCAL SOLUTION)

6a). SOLUTION TO $\dot{x}=\lambda x \quad x(0)=x_0: \quad x(t)=x_0 e^{\lambda t}$

$\Rightarrow$ VARIATION OF CONSTANT ANSATZ

$$x(t)=a(t)e^{\lambda t} \quad \text{SOLVES} \quad \dot{x}(t)=\lambda x(t)+b(t,x(t))$$

$$\dot{x}(t)=\dot{a}(t)e^{\lambda t}+a(t)e^{\lambda t}\lambda=\lambda a(t)e^{\lambda t}+b(t,a(t)e^{\lambda t})$$

$$\Rightarrow \dot{a}(t)e^{\lambda t}=b(t,x(t))$$

$$\Rightarrow \dot{a}(t)=e^{-\lambda t}b(t,x(t))$$

$$\Rightarrow a(t)=C+\int_{t_0}^{t}e^{-\lambda s}b(s,x(s))ds$$

$$\Rightarrow x(t)=Ce^{\lambda t}+\int_0^t e^{\lambda(t-s)}b(s,x(s))ds \quad \text{AND} \quad x(0)=C=x_0$$

$$\Rightarrow x(t)=x_0e^{\lambda t}+\int_0^t e^{\lambda(t-s)}b(s,x(s))ds.$$

$$\Rightarrow |x(t)|\le|x_0e^{\lambda t}|+\int_0^t e^{\lambda(t-s)}|b(s,x(s))|ds$$

$$\le|x_0e^{\lambda t}|+\int_0^t e^{\lambda(t-s)}\gamma(s)|x(s)|ds$$

$$\Rightarrow |x(t)e^{-\lambda t}|\le|x_0|+\int_0^t\gamma(s)|x(s)e^{-\lambda s}|ds.$$

$$\Rightarrow |x(t)e^{-\lambda t}|\le|x_0|e^{\int_0^t\gamma(s)ds} \qquad \text{BY GRONWALL}$$

$$|x(t)|\le|x_0|\,e^{\lambda t+\int_0^t\gamma(s)ds}$$

b) A SOLUTION $x^*(t)$ IS ASYMPTOTICALLY STABLE IF THERE EXISTS $\eta>0$ ST

$$\|x^*(t_0)-x(t_0)\|<\eta \Rightarrow \lim_{t\to\infty}\|x(t)-x^*(t)\|=0.$$

IF WE ASSUME THAT $\int_0^\infty \gamma(s)ds=D<\infty$ AND $\lambda<0$

$$\Rightarrow |e^{\lambda t+\int_0^t\gamma(s)ds}|\le e^{D+\lambda t}\to 0 \quad \text{AS } t\to\infty$$

$$\Rightarrow |x(t)|\le|x_0|e^{D+\lambda t}\to 0 \quad \text{AS } t\to\infty \quad \forall x_0$$

$$\Rightarrow |x(t)-x^*(t)|\le|x(t)|+|x^*(t)|\le(|x_0|+|x_0^*|)e^{D+\lambda t}\to 0 \quad \text{AS } t\to\infty$$

LET $\eta>0$ THEN $|x_0-x_0^*|<\eta \Rightarrow \lim_{t\to\infty}|x(t)-x^*(t)|=0.$

(η ARBITRARY!)

7) LET $z(t)$ BE THE SOLUTION TO $\dot{\vec{z}}(t) = D z(t)$.

DEFINE $V(z) = -\lambda z_1^2 - \mu z_2^2 \Rightarrow \dot{V}(z) = -2\lambda z_1 \dot{z}_1 - 2\mu z_2 \dot{z}_2 = -2\lambda^2 z_1^2 - 2\mu^2 \dot{z}_2 \leq -2\mu(\lambda z_1^2 + \mu z_2^2) = +2\mu V(z)$.

$V(z)$ HAS AS LEVEL SETS ELLIPSES.

$\dot{V}(z) \leq 2\mu V(z) \Rightarrow V(z(t)) \leq V(z(t_0)) e^{2\mu(t-t_0)} \to 0$ AS $t \to \infty$.

$V(z(t)) = 0 \Rightarrow z(t) = 0$

$\vec{x}(t) = P\vec{z}(t)$ SOLVES $\dot{\vec{x}}(t) = A\vec{x}(t)$

LET $U(\vec{x}) = V(P^{-1}\vec{x}(t)) = V(\vec{z}(t))$, ... CONTINUOUS

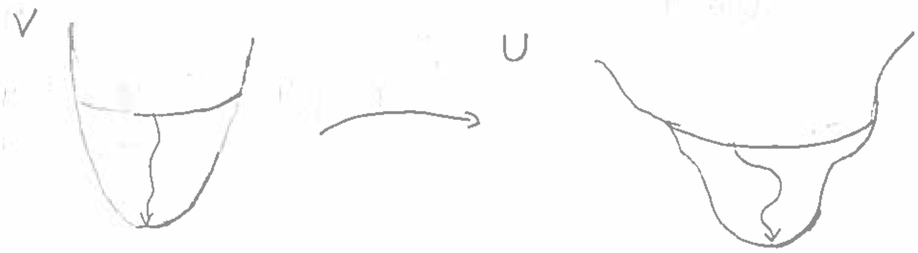
NOTE THAT $U(\vec{x})$ IS A STRONG LIAPUNOV FCT.

$$U(\vec{x}) = V(P^{-1}\vec{x}) = V(\vec{z}) > 0 \quad \forall \vec{x} \neq \vec{0}$$

$$U(\vec{0}) = V(\vec{0}) = \vec{0}$$

$$\dot{U}(\vec{x}) = \dot{V}(\vec{z}) < 0 \quad \forall \vec{x} \neq \vec{0}$$

THE LEVEL SETS OF V ARE MAPPED TO THE LEVEL SETS OF U.



$\Rightarrow$ SUFFICES TO SHOW THAT $\lim\limits_{t\to\infty} U(\vec{x}(t)) = 0$.

$$|U(\vec{x}(t))| = |V(P^{-1}\vec{x}(t))| = |V(z(t))| \leq |V(z(t_0))| e^{2\mu(t-t_0)} = |V(P^{-1}\vec{x}(t_0))| e^{2\mu(t-t_0)} = |U(x(t_0))| e^{2\mu(t-t_0)} \to 0$$

AS $t \to \infty$.