1) $\begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leadsto A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$

EIGENVALUES OF A: $(-\lambda)(3-\lambda) + 2 = 0$

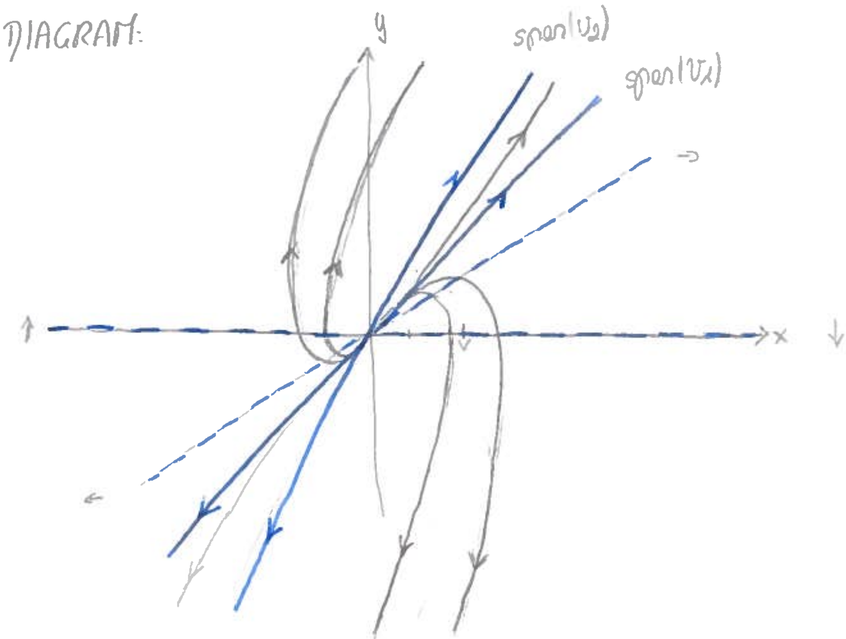
$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = \frac{3}{2} \pm \frac{1}{2} = \begin{cases} 1 \\ 2 \end{cases}$$

EIGENVECTORS: $\lambda_1 = 1$: $-x + y = 0 \leadsto v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 2$: $-2x + y = 0 \leadsto v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

PHASE DIAGRAM:



ISOCLINES: $\dot{x} = 0$: $\Rightarrow y = 0 \Rightarrow$ x-axis

$\dot{y} = 0 \Rightarrow -2x + 3y = 0 \Rightarrow \text{span}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix}$

2) LET $(x(t), y(t))$ BE A SOL TO (1), IE $\dot{x}(t) = y(t)$, $\dot{y}(t) = a x(t) + b y(t)$

$$\Rightarrow \dot{\tilde{x}}(t) = -\dot{x}(-t) = -y(-t) = \tilde{y}(t)$$

$$\dot{\tilde{y}}(t) = \dot{y}(-t) = a x(-t) + b y(-t) = a\tilde{x}(t) - b\tilde{y}(t) \overset{?}{=} a\tilde{x}(t) + b\tilde{y}(t).$$

$\Rightarrow a \neq 0$ & $b = 0$, IE $\begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \underbrace{\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}}_{=A} \begin{pmatrix} x \\ y \end{pmatrix}$

$\det(A) = -a \neq 0 \Rightarrow (0,0)$ ONLY EP

EIGENVALUES OF A: $(-\lambda)^2 - a = 0$

$\lambda^2 = a$

$\lambda_{12} = \pm\sqrt{a}$

- $a>0$: $(0,0)$ IS A SADDLE $\Rightarrow$ UNSTABLE
- $a<0$: $(0,0)$ IS A CENTRE $\Rightarrow$ (LIAPUNOV) STABLE BUT NOT ASYMPTOTICALLY STABLE.

3) a) EP: $\left.\begin{array}{l}\dot{x}=0 \Rightarrow y=0 \\ \dot{y}=0 \Rightarrow (x-2e^{y})(x+e^{-y})=0\end{array}\right\} \Rightarrow \left.\begin{array}{l} y=0 \\ (x-2)(x+1)=0\end{array}\right\} \Rightarrow \begin{array}{l}(-1,0)\\(2,0)\end{array}$

LIN: $$J_{(x,y)} = \begin{pmatrix} 0 & \frac{7}{3} \\ (x+e^{-y})+(x-2e^{y}) & -2e^{y}(x+e^{-y}) - e^{-y}(x-2e^{y}) \end{pmatrix}$$

$$J_{(-1,0)} = \begin{pmatrix} 0 & \frac{7}{3} \\ (-1+1)+(-1-2) & -2(-1+1) - 1(-1-2) \end{pmatrix} = \begin{pmatrix} 0 & \frac{7}{3} \\ -3 & 3 \end{pmatrix}$$

EIGENVALUES: $(-\lambda)(3-\lambda)+7=0$

$\lambda^2 - 3\lambda + 7 = 0$

$\lambda_{12} = \frac{3}{2} \pm \sqrt{\underbrace{\frac{9}{4} - 7}_{<0}}$ $\rightsquigarrow$ UNSTABLE SPIRAL

$$J_{(2,0)} = \begin{pmatrix} 0 & \frac{7}{3} \\ (2+1)+(2-2) & -2(2+1) - 1(2-2) \end{pmatrix} = \begin{pmatrix} 0 & \frac{7}{3} \\ 3 & -6 \end{pmatrix}$$

EIGENVALUES: $-\lambda(-6-\lambda) - 7 = 0$

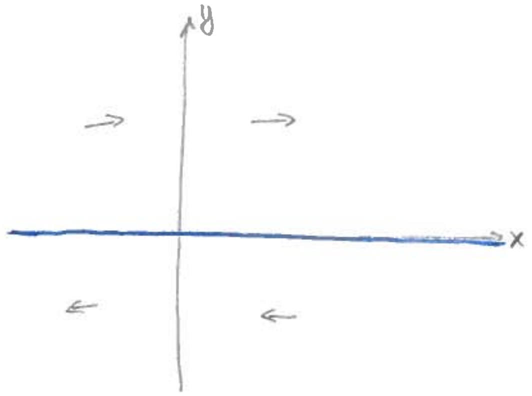
$\lambda^2 + 6\lambda - 7 = 0$

$\lambda_{12} = -3 \pm \sqrt{9+7} = -3 \pm 4 \begin{array}{l} \nearrow 1 \\ \searrow -7 \end{array}$ $\rightsquigarrow$ SADDLE:

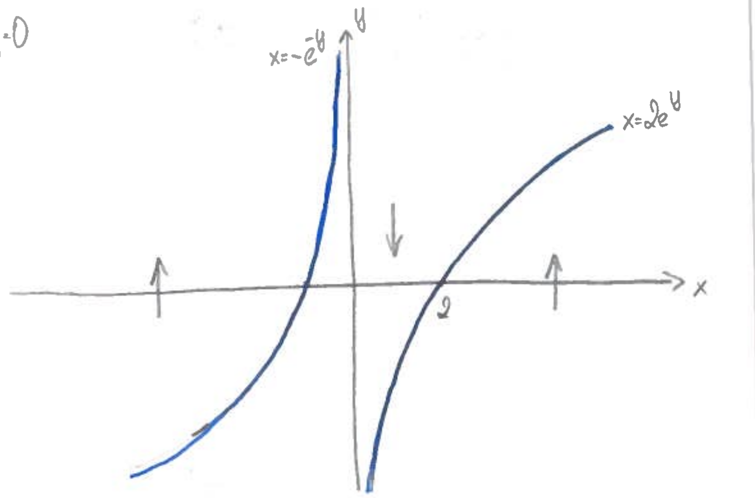
EIGENVECTORS: $\lambda_1 = 1$: $v_1 = \begin{pmatrix} \frac{7}{3} \\ 1 \end{pmatrix} \sim \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

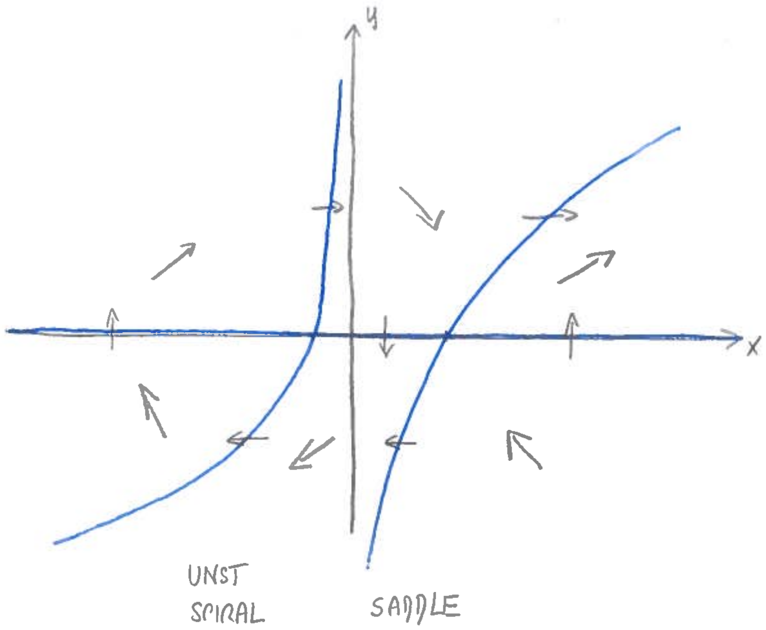
$\lambda_2 = -7$: $v_1 = \begin{pmatrix} \frac{7}{3} \\ -7 \end{pmatrix} \sim \begin{pmatrix} 7 \\ -21 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

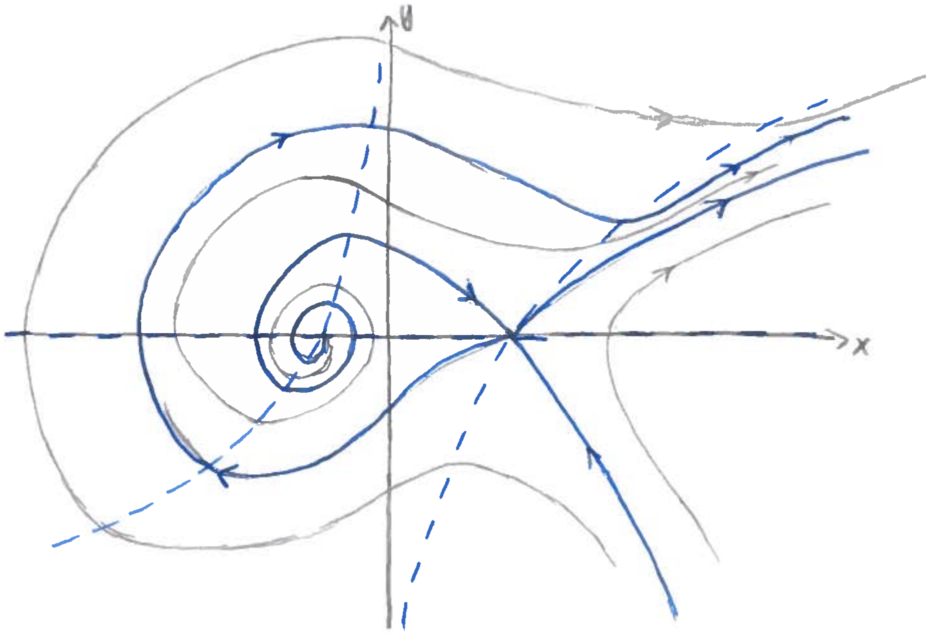
b) ISOCLINES: $\dot{x}=0$: $y=0$



$\dot{y}=0$







4e) $J_{(x,y)} = \begin{pmatrix} 2g(x,y)+2(x-1)g_x(x,y) & 2(x-1)g_y(x,y)+1 \\ -1+2yg_x(x,y) & 2g(x,y)+2yg_y(x,y) \end{pmatrix}$

$\rightarrow J_{(1,0)} = \begin{pmatrix} 2g(1,0) & 1 \\ -1 & 2g(1,0) \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ -1 & 6 \end{pmatrix}$

EIGENVALUES $6 \pm i$ $\Rightarrow$ UNSTABLE SPIRAL $\rightsquigarrow$ UNSTABLE EP

b) WANT TO USE POINCARÉ - BENDIXSON:

LET $V(x,y) = (x-1)^2 + y^2$

$\Rightarrow$ V HAS A STRONG MINIMUM AT (1,0).

$$\begin{aligned}(V(x,y))^{\cdot} &= 2(x-1)\dot{x} + 2y\dot{y} \\ &= 4(x-1)^2 g(x,y) + 2(x-1)y - 2(x-1)y + 4y^2 g(x,y). \\ &= 4\big(\underbrace{(x-1)^2+y^2}_{>0 \text{ IF } (x,y)\neq(1,0).}\big)g(x,y)\end{aligned}$$

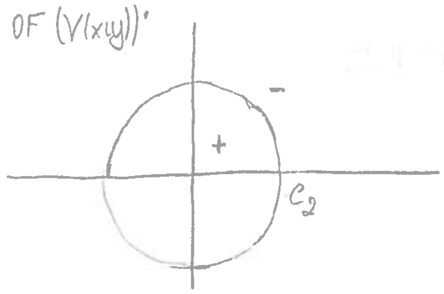
$\Rightarrow$ SIGN ONLY DEPENDENT ON THE SIGN OF $g(x,y)$

$x^2+y^2=4$ ... CIRCLE WITH RADIUS 2 CENTERED AT (0,0) (DENOTED C_2)

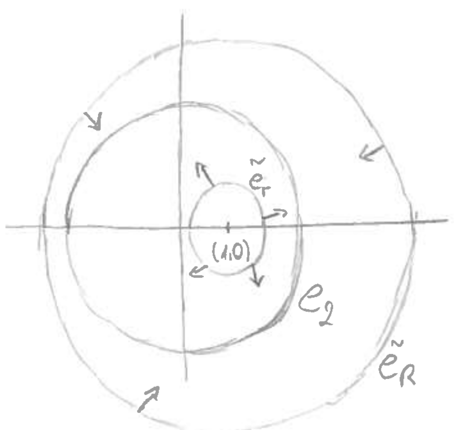
$\Rightarrow$ $g(x,y)>0$ IF (x,y) LIES INSIDE THIS CIRCLE

$g(x,y)<0$ IF (x,y) LIES OUTSIDE THIS CIRCLE.

$\Rightarrow$ SIGN OF $(V(x,y))'$



LEVEL SETS OF V: CIRCLES CENTERED AT (1,0).



$\Rightarrow$ $\tilde{C}_r = \{(x,y) \mid (x-1)^2+y^2=r^2\}$

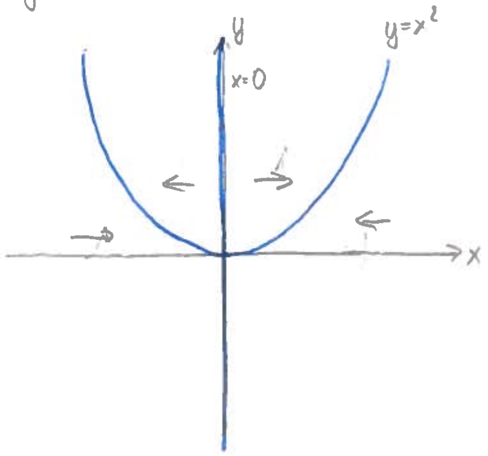
- IF $0<r<1$
  $\Rightarrow$ $\tilde{C}_r$ LIES INSIDE C_2
- IF $3<R$
  $\Rightarrow$ $\tilde{C}_R$ LIES OUTSIDE C_2.

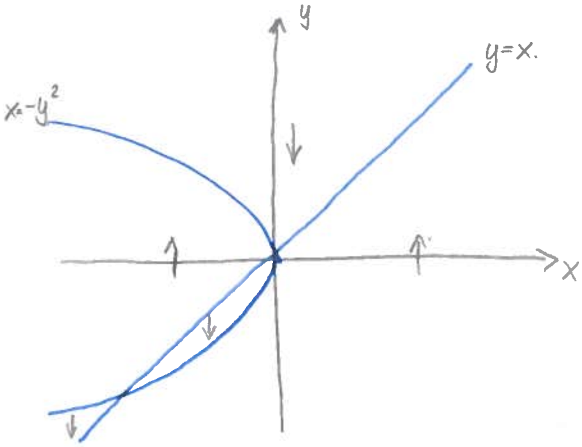
$\Rightarrow$ ALL PHASE PATHS STARTING INSIDE THE REGION BOUNDED BY $\tilde{C}_r$ & $\tilde{C}_R$ REMAIN INSIDE & NO EP INSIDE (TO CHECK)

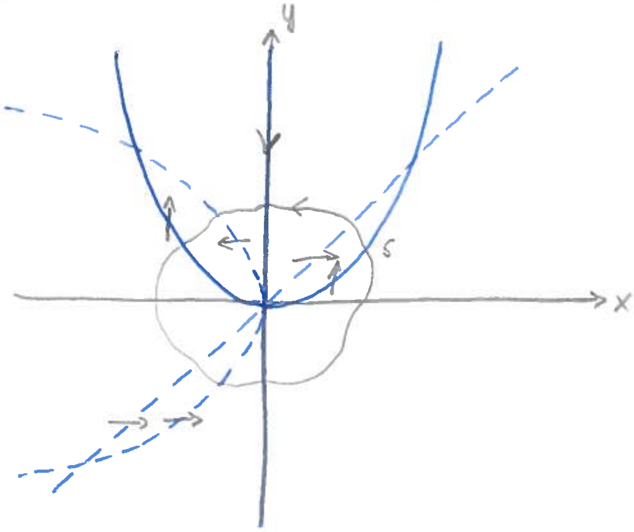
$\Rightarrow$ POINCARÉ - BENDIXSON: THERE EXISTS A NON-CONSTANT PERIODIC SOLUTION

EP: $\left.\begin{array}{l}\dot{x}=0 \Rightarrow y=-2(x-1)g(x,y) \\ \dot{y}=0 \Rightarrow -(x-1)+2yg(x,y)=0\end{array}\right\} \Rightarrow \left.\begin{array}{l}y=-2(x-1)g(x,y) \\ -(x-1)-2(x-1)g^2(x,y)=0\end{array}\right\} \Rightarrow \begin{array}{l}x=1 \\ y=0.\end{array}$

5a) $\dot{x} = x(y-x^2)$ $\qquad\qquad$ $\dot{y} = (x-y)(x+y^2)$







INDEX (STARTING AT S) $\quad \uparrow \rightsquigarrow \rightarrow \rightsquigarrow \downarrow \rightsquigarrow \leftarrow \rightsquigarrow \uparrow \rightsquigarrow \rightarrow \rightsquigarrow \rightarrow \rightsquigarrow \uparrow$

INDEX: -1.

b) 3 EP: $(0,0)$, $(1,1)$ & $(-1,1)$

$$\begin{cases} \dot{x}=0 \\ \dot{y}=0 \end{cases} \Rightarrow \begin{cases} x=0 \ \vee\ y=x^2 \\ (x-y)(x+y^2)=0 \end{cases} \Rightarrow$$

IF $x=0 \Rightarrow y=0$

IF $x^2=y \Rightarrow$ EITHER: $x-x^2=0 \rightsquigarrow x(1-x)=0 \rightarrow x=0,1$

OR $\quad x+x^4=0 \rightsquigarrow x(1+x^3)=0 \Rightarrow x=0,-1$. $\lrcorner$

$\mathcal{C}$... CIRCLE CENTERED AT $\left(\frac{1}{2},\frac{1}{2}\right)$ WITH RADIUS 1

$\Rightarrow$ $(0,0)$ ... INSIDE $\mathcal{C}$

$(1,1)$ ... INSIDE $\mathcal{C}$

$(-1,1)$ ... OUTSIDE $\mathcal{C}$

$\Rightarrow I_{\mathcal{C}} = I_{(0,0)} + I_{(1,1)} = -1+1=0$ $\qquad$ SINCE

$$J(x,y) = \begin{pmatrix} (y-x^2)-2x^2 & x \\ (x+y^2)+(x-y) & -(x+y^2)+2y(x-y) \end{pmatrix}$$

$$J_{(1,1)} = \begin{pmatrix} -2 & 1 \\ 2 & -2 \end{pmatrix} \rightsquigarrow \text{EIGENVALUES:} \quad (2+\lambda)^2 - 2 = 0$$

$$\lambda^2 + 4\lambda + 4 - 2 = 0$$

$$\lambda^2 + 4\lambda + 2 = 0$$

$$\lambda_{12} = -2 \pm \sqrt{4-2} \quad \text{.. STABLE NODE}$$

$$\Rightarrow I_{(1,1)} = 1$$

6) $\dot{x} = -2x + (1+2t^2)e^{(1-t)^2}, \quad x(1) = 1$

HOMOGENEOUS EQUATION: $\dot{x} = -2x$.

SOL. $x(t) = ce^{-2t}$. (c.. CONSTANT)

INHOMOGENEOUS EQUATION: $\dot{x} = -2x + (1+2t^2)e^{(1-t)^2}$

ANSATZ: $x(t) = c(t)e^{-2t}$

$$\Rightarrow \dot{x}(t) = \dot{c}(t)e^{-2t} - 2c(t)e^{-2t} = \dot{c}(t)e^{-2t} - 2x(t) \overset{?}{=} -2x(t) + (1+2t^2)e^{(1-t)^2}$$

$$\Rightarrow \dot{c}(t)e^{-2t} = (1+2t^2)e^{(1-t)^2}$$

$$\dot{c}(t) = (1+2t^2)e^{1+t^2}$$

$$c(t) = c(1) + \int_1^t (1+2s^2)e^{1+s^2}\,ds$$

$$\int_1^t 2s^2 e^{1+s^2}\,ds = \int_1^t s\, 2s e^{1+s^2}\,ds = se^{1+s^2}\Big|_1^t - \int_1^t e^{1+s^2}\,ds$$

$$= te^{1+t^2} - e^2 - \int_1^t e^{1+s^2}\,ds$$

$$\Rightarrow c(t) = c(1) + te^{1+t^2} - e^2$$

$$\Rightarrow x(t) = c(t)e^{-2t} = \left(c(1) + te^{1+t^2} - e^2\right)e^{-2t}$$

$$x(1) = 1 \Rightarrow \quad 1 = \left(c(1) + e^2 - e^2\right)e^{-2} \Rightarrow c(1) = e^2$$

$$\Rightarrow x(t) = te^{(1-t)^2}$$

7) $\begin{array}{l} \dot{x} = 1 \\ \dot{y} = 3 \end{array} \Rightarrow \begin{array}{l} x(t) = x(t_0) + (t-t_0) \\ y(t) = y(t_0) + 3(t-t_0) \end{array}$

$(x_1(t), y_1(t))$ IS (LIAPUNOV) STABLE FOR $t \geq t_0$ IF TO EVERY $\varepsilon > 0$ THERE EXISTS $\delta > 0$ ST $(x_2(t) - x_1(t))^2 + (y_2(t) - y_1(t))^2 < \varepsilon^2$ FOR ALL SOLUTIONS $(x_2(t), y_2(t))$ ST $(x_2(t_0) - x_1(t_0))^2 + (y_2(t_0) - y_1(t_0))^2 < \delta$ AND ALL $t \geq t_0$

IN OUR CASE:

$$(x_2(t)-x_1(t))^2+(y_2(t)-y_1(t))^2=(x_2(t_0)-x_1(t_0))^2+(y_2(t_0)-y_1(t_0))^2$$

$\Rightarrow$ WE CAN CHOOSE $\varepsilon=\delta$ AND ANY t_0.

$\Rightarrow$ ALL SOL. ARE (LIAPUNOV) STABLE FOR $t\geq t_0$.