TMA4165 Differential equations and dynamical systems 2019-08-06

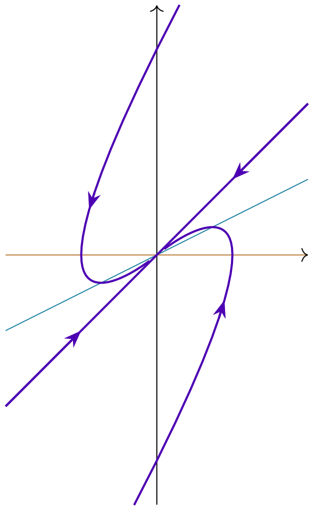
Solution

Problem 1

a. The system is linear, with matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix},$$

which has trace $p = -2$ and determinant $q = 1$. The characteristic polynomial is $\lambda^2 - p\lambda + q = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$, so there is only one eigenvalue $\lambda = -1$. The corresponding eigenspace is spanned by $(1, 1)$. It is helpful to note the nullclines, as shown in the picture: $\dot{x} = 0 \iff y = 0$ and $\dot{y} = 0 \iff x = 2y$. Since the system is linear, the phase diagram is scale invariant, so there is no need to draw more curves than shown.



b. We find $r\dot{r} = x\dot{x} + y\dot{y} = -2y^2$ and $r^2\dot{\theta} = x\dot{y} - y\dot{x} = x^2 - 2xy + y^2 = (x-y)^2$. Substituting in $x = r\cos\theta$ and $y = r\sin\theta$ we end up with

$$\dot{r} = -2r\sin^2\theta, \quad \dot{\theta} = (\cos\theta - \sin\theta)^2.$$

Note that $\dot{\theta} = 0 \iff \cos\theta = \sin\theta \iff \theta = (k + \frac{1}{2})\pi$ for some integer k. This corresponds to the invariant line $y = x$. For all other values, θ grows from $(k - \frac{1}{2})\pi$ (at $t = -\infty$) to $(k + \frac{1}{2})\pi$ (at $t = +\infty$), which matches the curved paths in the picture. Also $\dot{r} < 0$ except when $\theta = k\pi$, i.e., on the x-axis, where $\dot{r} = 0$.

Problem 2

The divergence of the associated vector field is

$$5 + 2xy - y^2 - 3 - x^2 - 2xy = 2 - x^2 - y^2,$$

which is positive in the specified region.

Problem 3

We want to find a strong Lyapunov function. A good start is to compute

$$\begin{aligned} x\dot{x} &= -2x^2 - 2xy + 2xyz, \\ y\dot{y} &= xy - y^2 - xyz - xyz^2, \\ z\dot{z} &= 3xyz^2 - z^4. \end{aligned}$$

Next, look for a linear combination of these with positive coefficients resulting in something negative. This results in

$$3x\dot{x} + 6y\dot{y} + 2z\dot{z} = -6x^2 - 6y^2 - 2z^4 < 0 \qquad \text{for } (x,y,z) \neq (0,0,0),$$

so $3x^2 + 6y^2 + 2z^2$ is indeed a strong Lyapunov function.

In this case, we were fortunate to get rid of all cross terms (like xy), though we have inequalities to deal with those if they are not too large. In more complicated cases, cross terms in the Lyapunov functions may be needed as well.

Problem 4

a. To show that the system is Hamiltonian, we may compute the divergence of the associated vector field $(X, Y) = (x^2y + 2y^3, -2x^3 - xy^2)$. The result is $2xy - 2xy = 0$, so the system is indeed Hamiltonian.

To find a Hamiltonian, we must solve the equations $H_y = X$, $H_x = -Y$ (where the subscripts denote partial derivatives). The first of these equations is $H_y = x^2y + 2y^3$, which we integrate as $H = \frac{1}{2}(x^2y^2 + y^4) + C(x)$ where the integration "constant" $C(x)$ may depend on x. Plugging this into the second equation, $H_x = 2x^3 + xy^2$, yields $xy^2 + C'(x) = 2x^3 + xy^2$. Canceling terms yields $C'(x) = 2x^3$, with the solution $C(x) = \frac{1}{2}x^4$ (plus an integration constant that we don't care about). This results in $H(x) = \frac{1}{2}(x^4 + x^2y^2 + y^4)$.

We did not really need to do the first part, as finding the Hamiltonian certainly proves that it exists!

b. Writing the system in the form $\dot{x} = H_y + (1 - x^2 - y^2)x$, $\dot{y} = -H_x + (1 - x^2 - y^2)y$, we find

$$\begin{aligned} \dot{H} &= H_x\big(H_y + (1 - x^2 - y^2)x\big) + H_y\big(-H_x + (1 - x^2 - y^2)y\big) \\ &= (xH_x + yH_y)(1 - x^2 - y^2) \\ &= 2(x^4 + x^2y^2 + y^4)(1 - x^2 - y^2). \end{aligned}$$

Thus $\dot{H} > 0$ for $0 < x^2 + y^2 < 1$, and $\dot{H} < 0$ for $x^2 + y^2 > 1$.

Now on one hand, $H(x, y) \le \frac{1}{2}(x^2 + y^2)^2$, so $H(x, y) > \frac{1}{2}$ implies $x^2 + y^2 > 1$, and therefore $\dot{H} < 0$.

On the other hand, $H(x, y) = \frac{3}{8}(x^2 + y^2)^2 + \frac{1}{8}(x^2 - y^2)^2 \ge \frac{3}{8}(x^2 + y^2)^2$, so $H(x, y) < \frac{3}{8}$ implies $x^2 + y^2 < 1$, and therefore $\dot{H} > 0$.

It follows that the compact set $\{\, (x, y) \mid \frac{3}{8} \le H(x, y) \le \frac{1}{2} \,\}$ is forward invariant. There are no equilibrium points in this set, for at such an equilibrium point, first $1 - x^2 - y^2 = 0$ (since $\dot{H} = 0$), and then also $x^2y + 2y^3 = 0$ and $-2x^3 - xy^2 = 0$. Writing these as $(x^2 + 2y^2)y = 0$ and $(2x^2 + y^2)x = 0$ makes it obvious that $(0, 0)$ is the only equilibrium point.

The Poincaré–Bendixsson theorem now shows the existence of a periodic path.

Problem 5

a. The reflection through the line $x = y$ maps (x, y) to (y, x). If $\big(x(t), y(t)\big)$ is a solution, then so is $\big(y(t), x(t)\big)$. Thus one phase path is mapped to the other, with orientations preserved.

The reflection through the line $x + y = 0$ maps (x, y) to $(-y, -x)$. If $\big(x(t), y(t)\big)$ is a solution, then so is $\big(-y(-t), -x(-t)\big)$. Thus one phase path is mapped to the other, with orientations reversed.

b. The equilibrium points are $(0, 0)$, $(1, 1)$, $(-1, -1)$, $(1, -1)$ and $(-1, 1)$.

The linearization matrix at any equilibrium point (x, y) is

$$A = \begin{pmatrix} -4x^3 & 2y \\ 2x & -4y^3 \end{pmatrix}$$

At $(1, 1)$, we find

$$A = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}, \quad p = \operatorname{tr} A = -8 < 0, \quad q = \det A = 12 > 0, \quad p^2 - 4q > 0,$$

so $(1, 1)$ is a stable node. In particular, it is asymptotically stable.

The equilibrium at $(-1, -1)$ is the image of $(1, 1)$ by the orientation reversing reflection through the line $x + y = 0$, so $(-1, -1)$ is an unstable node.

At $(1,-1)$, we find

$$A = \begin{pmatrix} -4 & -2 \\ 2 & 4 \end{pmatrix}, \quad p = \operatorname{tr} A = 0, \quad q = \det A = -12 < 0,$$

so $(1,-1)$ is a saddle point, and therefore unstable.

The equilibrium at $(-1,1)$ is the image of $(1,-1)$ by the orientation preserving reflection through the line $x = y$, so $(-1,1)$ is a saddle point, too (and unstable).

The linearization of the equilibrium at $(0,0)$ is trivial ($A = 0$), so it has no simple classification. We can note that the diagonal $x = y$ is invariant, with solutions satisfying $\dot{x} = x^2 - x^4$ and $y = x$. In particular, since $\dot{x} > 0$ for $0 < x < 1$, the origin is unstable.

c. The equilibria at $(1,1)$ and $(-1,-1)$, being nodes, have index 1. The equilibria at $(1,-1)$ and $(-1,1)$, being saddles, have index -1. The index at $(0,0)$ is easily found to be zero by direct examination. For example, when $x^2 + y^2$ is small and non-zero, $(\dot{x}, \dot{y}) = (y^2 - x^4, x^2 - y^4)$ never enters the third quadrant, since $y^2 - x^4 < 0$ and $x^2 - y^4 < 0$ imply $x^2 < y^4 < x^8$, so $|x| > 1$. Thus the index of a small closed curve around the origin must be zero. (Of course, the more convention argument tracking the direction of arrows works just as well.)

d. The linearization matrix at $(1,1)$ has characteristic polynomial $\lambda^2 - p\lambda + q = \lambda^2 + 8\lambda + 12 = (\lambda+4)^2 - 4 = (\lambda+2)(\lambda+6)$, so the eigenvalues are -2 and -6. For the eigenvalue -6 we find the eigenvector $(-1,1)$, and for the eigenvalue -2 we find the eigenvector $(1,)$.

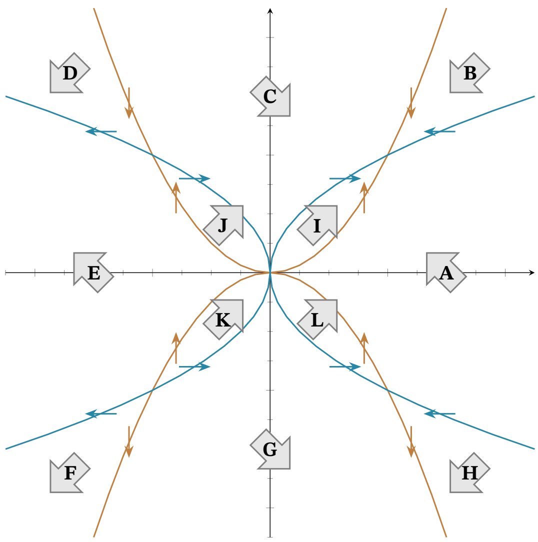
Due to symmetry, we find that the linearization at $(-1,-1)$ has eigenvalues 6 and 2, with eigenvectors $(-1,1)$ and $(-1,1)$ respectively.

The linearization matrix at $(1,-1)$ has characteristic polynomial $\lambda^2 - p\lambda + q = \lambda^2 - 12$, so the eigenvalues are $\pm 2\sqrt{3}$. Corresponding eigenvectors are $(1, -2 \mp \sqrt{3})$.

It may be worth noting that $(-2-\sqrt{3})(-2+\sqrt{3}) = 1$, so that the two eigenspaces are symmetric with respect to the line $x + y = 1$, as they should due to the symmetry of the system.

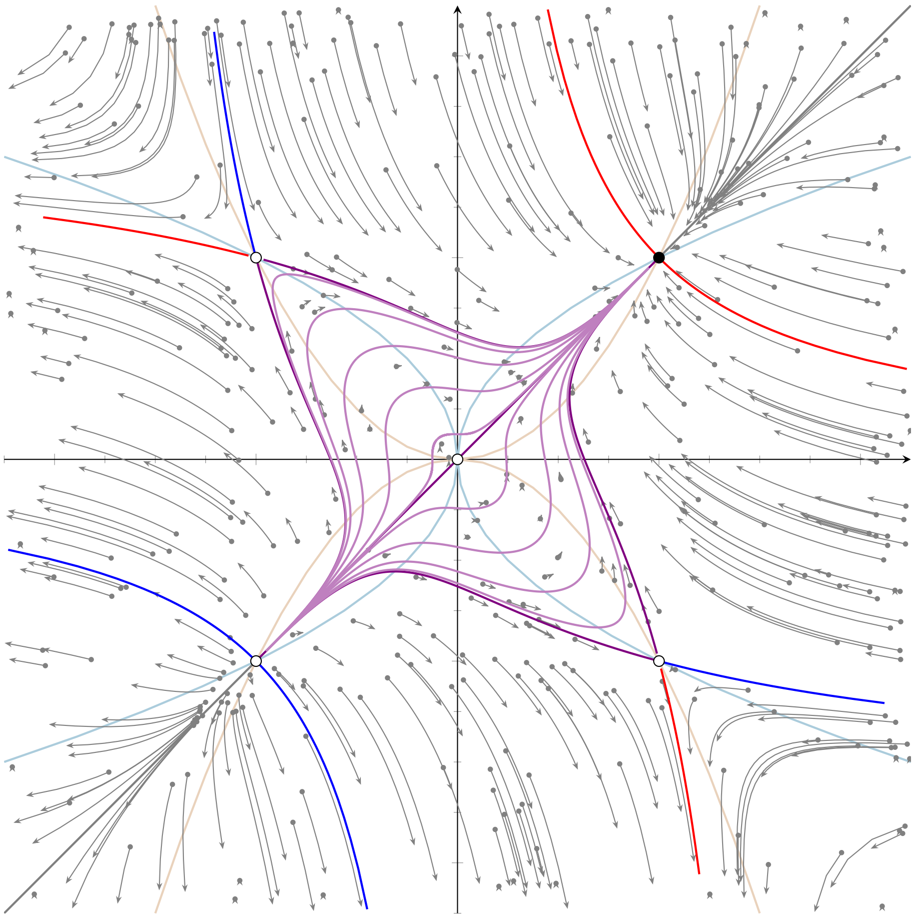
Thanks to the symmetry in the line $x = y$, the equilibrium at $(-1,1)$ also has eigenvalues $\pm 2\sqrt{3}$, with eigenvectors $(-2 \mp \sqrt{3}, 1)$ (which may be rescaled to $(1, -2 \pm \sqrt{3})$).

e. To get started, here is a picture showing the x- and y-nullclines, defined respectively by $\dot{x} = 0$ $(y = \pm x^2)$ and $\dot{y} = 0$ $(x = \pm y^2)$. The nullclines are equipped with arrows showing the direction of the flow at each segment. The nullclines divide the plane into twelve regions, named A–L, and the general direction of the flow is indicated by a fat arrow in each region.



From this diagram, it is clear that any phase path in region A must enter either region I or B, and approach the node at $(1,1)$ from there. (Except one orbit that approaces the node along the eigenvector $(1,-1)$.)

Similarly, any phase path passing through region L must come from the node at $(-1,-1)$ and then pass through regions K, G, L, A, I and end up at node $(1,1)$. Thus all these are heteroclinic orbits. A similar argument shows that two separatrices from the saddle point $(1,-1)$ are also heteroclinic orbits, with the other end at $(-1,-1)$ and $(1,1)$ respectively. Phase paths in region G on the "wrong" side of the separatric go to infinity as $t \to \infty$, and to $(-1,-1)$ as $t \to -\infty$. Symmetry takes care of the rest – i.e., we only need to argue the image below the diagonal $y = x$.



Here is a computer generated phase diagram. It is of course unrealistic (not to mention *unfair*) to expect every aspect to be present in a hand drawn version, but the more the merrier! The grey curve pieces are partial phase paths. They have a circle at the start, and an arrow at the end, thus indicating direction. Their lengths are roughly proportional to average flow speed. The stable node is shown as a filled black circle, while the four unstable equilibria are white circles. The purple paths are heteroclinic orbits, while the other separatrices at the saddle points are marked blue (approaching the saddlepoint) and red (leaving the saddlepoint). Separatrices at the nodes are similarly marked. They are tangent to the eigenvector corresponding to the eigenvalue furthest from zero; all other phase paths approaching the node do so tangentially to the other eigenvector.