TMA4165
Differential Equations
and Dynamical Systems
Spring 2012

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Solutions to exam spring 2012

1

a) We have the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}, \ A = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}.$$

Eigenvalues: $\det(A - \lambda I) = 0 \Rightarrow \lambda(\lambda - 2) - 3 = 0$

$$\Rightarrow \lambda_{\pm} = \frac{2 \pm \sqrt{4+12}}{2} = 1 \pm 2.$$

Eigenvectors: $0 = (A - \lambda I)\mathbf{r} = \begin{bmatrix} 1 \mp 2 & 1 \\ -3 & 1 \pm 2 \end{bmatrix} \mathbf{r}_{\pm}$, so take e.g.

$$\mathbf{r}_{+} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \mathbf{r}_{-} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

General solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathrm{e}^{3t} + C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \mathrm{e}^{-t}.$$

Phase diagram:

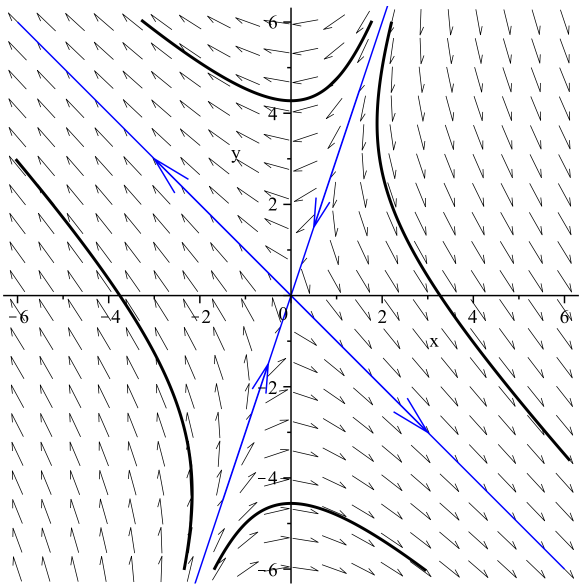

Figure 1: Phase diagram for $\dot{\mathbf{x}} = A\mathbf{x}$. The blue lines indicate asymptotes.

b) Note that
$$\Phi = [\mathbf{x}_1\,\mathbf{x}_2] = \mathrm{e}^{tA} \qquad \text{if} \qquad \dot{\Phi} = A\Phi,\ \Phi(0) = I.$$
or equivalently if
$$\begin{aligned}\dot{\mathbf{x}}_1 &= A\mathbf{x}_1,\ \mathbf{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\\ \dot{\mathbf{x}}_2 &= A\mathbf{x}_2,\ \mathbf{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.\end{aligned}$$
By a),
$$\mathbf{x}_i = C_1\mathbf{r}_+\mathrm{e}^{\lambda_+ t} + C_2\mathbf{r}_-\mathrm{e}^{\lambda_- t}$$
and hence
$$\begin{aligned}\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \mathbf{x}_1(0) = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 3 \end{bmatrix}\\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \mathbf{x}_2(0) = \bar{C}_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \bar{C}_2\begin{bmatrix} 1 \\ 3 \end{bmatrix}.\end{aligned}$$
We find that
$$1 = C_1 + C_2, \quad C_1 = 3C_2 \qquad \Rightarrow \qquad C_1 = \frac{3}{4}, \quad C_2 = \frac{1}{4},$$
and
$$\bar{C}_1 = -\bar{C}_2,\ 1 = -\bar{C}_1 + 3\bar{C}_2 \qquad \Rightarrow \qquad \bar{C}_1 = -\frac{1}{4},\ \bar{C}_2 = \frac{1}{4}.$$
Conclusion:
$$\mathrm{e}^{tA} = [\mathbf{x}_1\,\mathbf{x}_2] = \underline{\frac{1}{4}\begin{bmatrix} 3\mathrm{e}^{3t} + \mathrm{e}^{-t} & -\mathrm{e}^{3t} + \mathrm{e}^{-t} \\ -3\mathrm{e}^{3t} + 3\mathrm{e}^{-t} & \mathrm{e}^{3t} + 3\mathrm{e}^{-t} \end{bmatrix}}$$

2

We have
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = f(x,y) = \begin{bmatrix} x(1 + x^2 + y^4) - \mathrm{e}^y \\ y(1 + x^2 + y^4) + x \end{bmatrix}.$$
Since
$$\nabla \cdot f = f_x + f_y = (1 + x^2 + y^4) + 2x^2 + (1 + x^2 + y^4) + 4y^4 > 0 \quad \text{in} \quad \mathbb{R}^2,$$
<u>there are no closed trajectories/cycles</u> by Bendixson's negative criterion.

3

a) We consider

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = f(x,y) = \begin{bmatrix} x^2 - y \\ (x-y)(1-y) \end{bmatrix}$$

Equilibrium points:

$$\mathbf{0} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = f(x,y) \quad \Rightarrow \quad y = x^2 \quad \text{and} \quad (x = y \text{ or } y = 1)$$
$$\Rightarrow \quad \underline{(x,y) \in \{(0,0),\ (1,1),\ (-1,1)\}}.$$

Jacobian $Df(x,y) = \begin{bmatrix} 2x & -1 \\ 1-y & -1-x+2y \end{bmatrix}$.

Eigenvalues for $Df(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$:

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \qquad \Rightarrow \qquad \underline{(0,0) \text{ is a stable spiral point}}$$

Eigenvalues for $Df(1,1) = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ (on the diagonal!):

$$\lambda_1 = 2,\ \lambda_2 = 0 \qquad \Rightarrow \qquad \underline{(1,1) \text{ cannot be classified by linearization}}$$

Eigenvalues for $Df(-1,1) = \begin{bmatrix} -2 & -1 \\ 0 & 2 \end{bmatrix}$:

$$\lambda = \pm 2 \qquad \Rightarrow \qquad \underline{(-1,1) \text{ is a saddle point}}.$$

b) The circle S with center $(-1,0)$ and radius 2 contains the equilibrium points $(-1,1)$ and $(0,0)$, but not $(1,1)$ (and there are no other (finite) equilibrium points!). Hence, since the system is smooth, we have that

$$I_S = I_{(-1,1)} + I_{(0,0)} = 1 - 1 = \underline{0},$$

where we also used that the index of a saddle point is -1 and +1 for a spiral point.

4

Observe that the system is linear (in x and y) and can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \big(A + B(t)\big) \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \mathbf{b}(t)$$

where

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-t} \\ (1+t)\mathrm{e}^{-t} & 0 & 0 \end{bmatrix}, \quad \mathbf{b}(t) = \begin{bmatrix} -\mathrm{e}^{-t}\sin t \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix}.$$

A is in Jordan form with eigenvalues $\lambda = -1 \pm i$ and $\lambda = 0$, hence $\max_i \mathrm{Re}\lambda_i = 0$. But since all eigenvalues are distinct (the 0-eigenvalue is single), the general solution and fundamental matrix Φ_A of

$$\dot{\mathbf{x}} = A\mathbf{x}$$

will be bounded. Next, since $\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t < \infty$ and $\int_0^\infty (1+t)\mathrm{e}^{-t}\,\mathrm{d}t < \infty$ (integration by parts),

$$\int_0^\infty \|B(t)\|\,\mathrm{d}t \leq 2\int_0^\infty \max_{i,j} |B_{ij}(t)|\,\mathrm{d}t < \infty.$$

By the stability theory, since Φ_A is bounded and $\int_0^\infty \|B(t)\|\,\mathrm{d}t < \infty$, all solutions are bounded.

Observe that because of the 0-eigenvalue of A, we could not conclude *asymptotic* stability from this argument.

5

a) The flow $\varphi(t;x_0)$ is the unique solution of $\dot{x} = f(x)$ and $x(0) = x_0$. The phase trajectory through x_0 is then

$$\Gamma_{x_0} = \{\varphi(t;x_0) : t \in \mathbb{R}\},$$

and the ω-limit set of Γ_{x_0} is

$$\omega_{\Gamma_{x_0}} = \{z \in \mathbb{R}^2 : \exists t_n, \text{ such that } t_n \to \infty \text{ and } \varphi(t_n;x_0) \to z\}.$$

By Poincare-Bendixson's theorem, an ω-limit set in a closed bounded subset of $\mathbb{R}^2$ either contains equilibrium points (i.e. it is an equilibrium point or separatrix cycle) or it is a cycle (non-constant closed trajectory: limit cycle or "center cycle").

b) In both cases $\dot{r} > 0$ except for $r = 0$ and $r = 1$. Hence all (finite) ω-limit sets are subsets of $r = 0$ or $r = 1$.

The set $r = 0$ is a point – an equilibrium point (unstable node) in both cases.

The set $r = 1$ is a limit cycle since $\dot{\theta} = 1$ (no equilibrium points in r=1) in case (i), and each point in $r = 1$ is an (unstable) equilibrium point since $\dot{\theta} = 0$ in case (ii).

Since $\dot{r} > 0$ away from $r = 0$ and $r = 1$, the ω-limit sets are unstable.

6

By the inequality $2ab \leq a^2 + b^2$,

$$V = x^4 + x^2 y + 2y^2 \geq x^4 - \frac{1}{2}x^4 - \frac{1}{2}y^2 + 2y^2 = \frac{1}{2}x^4 + \frac{3}{2}y^2,$$

so $V(x,y) > 0$ for $(x,y) \neq (0,0)$ and $V(0,0) = 0$.

$$\begin{aligned}\dot V &= \nabla V \cdot f = V_x \cdot f_1 + V_y \cdot f_2\\ &= V_x(-V_x + V_y) + V_y(-V_x - V_y)\\ &= -V_x^2 - V_y^2 = -(4x^3 + 2xy)^2 - (x^2 + 4y)^2\end{aligned}$$

So $\dot V \geq 0$ and $\dot V = 0$ only when $x(4x^2 + 2y) = 0$ and $x^2 + 4y = 0$. If $x \neq 0$, then

$$4x^2 = -2y \quad \text{and} \quad x^2 = -4y \qquad \Rightarrow \qquad -16y = -2y \qquad \Rightarrow \qquad y = 0,$$

and then also $x = 0$. Hence, $\dot V > 0$ for $(x,y) \neq (0,0)$. Since $\dot V(0,0) = 0$ and V is smooth, the above computation implies that V is a strong Liapunov function and that $(0,0)$ is asymptotically stable.

7

We rewrite the equation as a system,

$$\begin{bmatrix} \dot x \\ \dot y \end{bmatrix} = f(x,y) = \begin{bmatrix} y \\ -\epsilon y - \sin x \end{bmatrix}, \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ v \end{bmatrix}.$$

We let the solutions of the two systems corresponding to $\epsilon = 0$ and $\epsilon > 0$ be $\big(x^0(t), y^0(t)\big)$ and $\big(x^\epsilon(t), y^\epsilon(t)\big)$ respectively, and set

$$\sigma(t) = \big(x^0(t) - x^\epsilon(t)\big)^2 + \big(y^0(t) - y^\epsilon(t)\big)^2.$$

By the equations (the systems!),

$$\begin{aligned}\dot\sigma &= 2\big(x^0(t) - x^\epsilon(t)\big)\big(\dot x^0(t) - \dot x^\epsilon(t)\big) + 2\big(y^0(t) - y^\epsilon(t)\big)\big(\dot y^0(t) - \dot y^\epsilon(t)\big)\\ &= 2\big(x^0(t) - x^\epsilon(t)\big)\big(y^0(t) - y^\epsilon(t)\big)\\ &\quad + 2\big(y^0(t) - y^\epsilon(t)\big)\big(- \sin x^0 - (-\epsilon y^\epsilon - \sin x^\epsilon)\big).\end{aligned}$$

By the definition of σ and since

$$|\sin x^0 - \sin x^\epsilon| \leq \max_x |\cos x||x^0 - x^\epsilon| \leq |x^0 - x^\epsilon|$$

we find that

$$\dot\sigma \leq 2\sqrt{\sigma}\sqrt{\sigma} + 2\sqrt{\sigma}\Big(|\epsilon||y^\epsilon| + \sqrt{\sigma}\Big).$$

Since $|y^\epsilon(t)| \leq M_t < \infty$ and $2ab \leq a^2 + b^2$,

$$\dot\sigma \leq 2\sigma + \sigma + |\epsilon|^2 M_t^2 + 2\sigma = 5\sigma + |\epsilon|^2 M_t^2.$$

Using an integrating factor, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{-5t}\sigma\Big) = \mathrm{e}^{-5t}(\dot\sigma - 5\sigma) \leq \mathrm{e}^{-5t} M_t^2 |\epsilon|^2,$$

and by integrating, we find that

$$\mathrm{e}^{-5t}\sigma(t) \leq \sigma(0) + M_t^2|\epsilon|^2 \int_0^t \mathrm{e}^{-5s}\ \mathrm{d}s$$

$(M_t \geq M_s, t \geq s)$ and hence since $\sigma(0) = 0$,

$$|x^0(t) - x^\epsilon(t)| \leq \sqrt{\sigma(t)} \leq M_t|\epsilon|\sqrt{\mathrm{e}^{5t}\int_0^t \mathrm{e}^{-5s}\ \mathrm{d}s}\ .$$