Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during the exam:
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Exam in TMA4165 Differential equations and dynamical systems

English
Thursday May 30. 2013
Time: 15:00 - 19:00

Permitted aids (Code D): Simple calculator (HP 30S or Citizen SR-270X).
Grades: 20. juni 2013.

All answers should be justified.

Problem 1 Given the system of differential equations:

$$\begin{cases} \dot{x} = (2 - x^2 - y^2)x \\ \dot{y} = x - y. \end{cases}$$

a) Find and classify all equilibrium points of the system.

b) Sketch the phase diagram of the system, with orientations.

Problem 2 Write up a system of differential equations with Hamiltonian

$$H(x, y) = \cos x \cos y.$$

Show that $(0, 0)$ and $(\frac{\pi}{2}, \frac{\pi}{2})$ are equilibrium points for the system.

Classify the equilibrium points $(0, 0)$ and $(\frac{\pi}{2}, \frac{\pi}{2})$.

Problem 3 Find the general solution of the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Problem 4 Compute the index of the origin for the system

$$\begin{cases} \dot{x} = x^2 - y^2 \\ \dot{y} = -2xy. \end{cases}$$

Problem 5 Determine if all the solutions of the following systems are stable, asymptotically stable, or unstable.

(i) $$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & -10 & 2 \\ 10 & 0 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(ii) $$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 10 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(iii) $$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -3 & 2 & \frac{1}{1+t^2} \\ 0 & -7 & 0 \\ e^{-t} & 0 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Problem 6 Let $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and $x : \mathbb{R} \to \mathbb{R}^n$ be a solution of

$$\dot{x}(t) = Ax(t) + b.$$

Define what it means for $x(t)$ to be stable (Liapunov stable).

Show that all solutions of the equation are stable if there is a fundamental matrix $\Phi : \mathbb{R} \to \mathbb{R}^{n\times n}$ for $\dot{x} = Ax$ such that

$$\|\Phi(t)\| \leq C < \infty \qquad \text{for all} \qquad t \geq 0.$$

Problem 7 Show that $(0,0)$ is an asymptotically stable equilibrium point for the system

$$\begin{cases} \dot{x} = 2(x^2 + 2y^2)y - x^3 \\ \dot{y} = -(x^2 + 2y^2)x - e^x y. \end{cases}$$

Show that the domain of attraction of $(0,0)$ is all of $\mathbb{R}^2$.

Problem 8 Determine whether or not the system has non-constant periodic solutions.

$$\begin{cases} \dot{x} = -y + x\sin(3x^2 + 2y^2) \\ \dot{y} = x + y\sin(3x^2 + 2y^2). \end{cases}$$

Problem 9 The velocity v of a relativistic electron in a constant electric field in one dimension satisfies

$$\begin{cases} \dot{v} = (1 - \epsilon^2 v^2)^{\frac{3}{2}}, \\ v(0) = v_0, \end{cases}$$

where $|\epsilon| < 1$ and $|v_0| < 1$.

Show, using a direct argument, that the initial value problem cannot have more than one solution.

Show that the solution exists for $|t| < 1$.

Hint: Show that the domain $|v| \leq \frac{1}{\epsilon}$ is invariant.