Norwegian University of Science and Technology
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TMA4165 Differential Equations and Dynamical Systems

Solutions to Exam - Spring 2013

1 **a)** To find the equilibrium points, we must ensure that $\dot{x}$ and $\dot{y}$ are zero simultaneously, this means that $x = y$ and $(2 - x^2 - y^2)x = 0$, so either $(x, y) = (0, 0)$ or $x^2 + y^2 = 2x^2 = 2$, so $x = y = \pm 1$.

Summing up, the equilibrium points are $(0, 0)$, $(1, 1)$ and $(-1, -1)$.

To classify these, we look at the Jacobian

$$Df(x, y) = \begin{bmatrix} 2 - x^2 - y^2 - 2x^2 & -2xy \\ 1 & -1 \end{bmatrix}.$$

At the origin, we have

$$Df(0, 0) = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}.$$

This matrix has eigenvalues 2 and -1, of opposite signs, so $(0, 0)$ is a saddle point.

At $(\pm 1, \pm 1)$, we have

$$Df(\pm 1, \pm 1) = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}.$$

The eigenvalues are the roots $\lambda_\pm = \frac{1}{2}\left(-3 \pm i\sqrt{7}\right)$, of the characteristic polynomial of the matrix:

$$\lambda^2 + 3\lambda + 4 = 0.$$

Since both eigenvalues are complex with negative real part, both $(1, 1)$ and $(-1, -1)$ are stable spirals.

b) In Figure 1, the phase diagram of the system is drawn. To do this by hand, one would first start by drawing the directions of the eigenvectors close to each equilibrium point.

Secondly, one looks at the isoclines

$$\dot{x} = 0 \Rightarrow x = 0 \text{ or } x^2 + y^2 = 2$$
$$\dot{y} = 0 \Rightarrow y = 0$$

Together with the classification of the equilibrium points above, this information is sufficient to sketch the diagram as done in the figure.

Lastly, to find the orientation, pick a point where f is particularly simple, for instance $f(0, 1) = (0, -1)$. This gives the orientation on the y-axis, and, by continuity, of the entire phase plane.

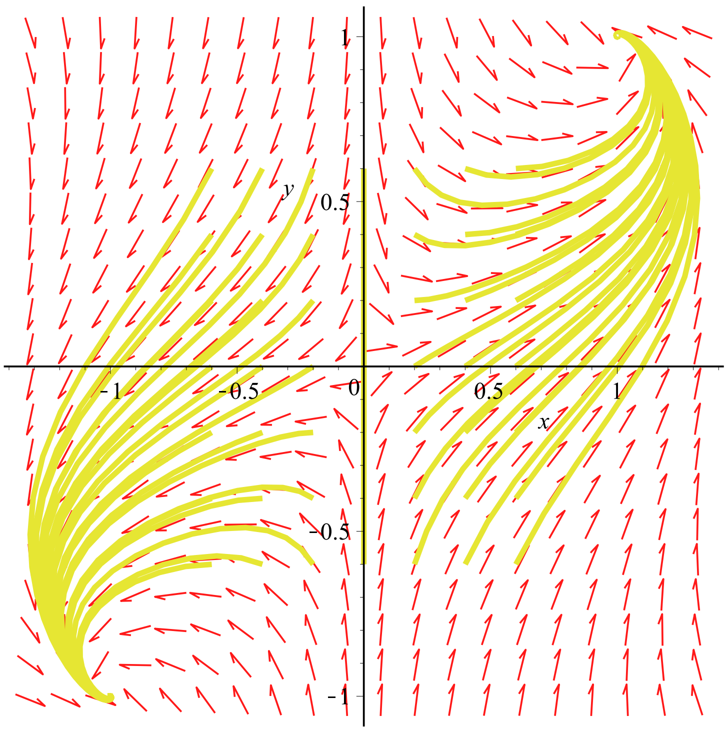

Figure 1: Phase diagram

$\boxed{2}$ Given H, one can find a dynamical system with H as its Hamiltonian as follows:

$$f(x,y) := \begin{cases} \dot{x} & = H_y = -\cos x \sin y \\ \dot{y} & = -H_x = \sin x \cos y \end{cases}.$$

To show that $(0,0)$ and $(\frac{\pi}{2}, \frac{\pi}{2})$ are equilibrium points, we simply plug those values into f:

$$f(0,0) = (-\cos 0 \sin 0, \sin 0 \cos 0) = (0,0),$$

and

$$f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \left(-\cos\frac{\pi}{2}\sin\frac{\pi}{2}, \sin\frac{\pi}{2}\cos\frac{\pi}{2}\right) = (0,0),$$

so both points are equilibrium points.

To classify these, we use the second derivative test.

$$D^2H = \begin{bmatrix} -\cos x \cos y & \sin x \sin y \\ \sin x \sin y & -\cos x \cos y \end{bmatrix}$$

Evaluating at each of the points gives

$$D^2H(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

with determinant $\det D^2H(0,0) = 1$, so $(0,0)$ is a center; and

$$D^2H\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

with determinant $\det D^2H\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = -1$, so $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ is a saddle.

$\boxed{3}$ As usual, we let A denote the matrix in question:

$$A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}.$$

We first find the eigenvalues λ of the matrix, satisfying $\det(A - \lambda I) = 0$:

$$(1-\lambda)(5-\lambda) + 4 \cdot 2 = 0 \Leftrightarrow \lambda^2 - 6\lambda + 13 = 0.$$

This equation has solutions

$$\lambda_\pm = \frac{6 \pm \sqrt{6^2 - 4 \cdot 13}}{2} = 3 \pm 2i.$$

The eigenvector r corresponding to λ_+ is found as follows:

$$\begin{aligned} &(A - \lambda_+ I)\, r = 0 \\ \Rightarrow &\begin{bmatrix} -2-2i & 2 \\ -4 & 2-2i \end{bmatrix} r = 0, \end{aligned}$$

from which we see that $r = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$ is *one* possible choice.

We can then write down a basis for solutions of the dynamical system:

$$\begin{aligned}
v_1 &= \Re\left(re^{\lambda_+ t}\right) = \Re\left(\left(\begin{bmatrix}1\\1\end{bmatrix} + i\begin{bmatrix}0\\1\end{bmatrix}\right) e^{3t}\left(\cos 2t + i\sin 2t\right)\right)\\
&= \begin{bmatrix}1\\1\end{bmatrix} e^{3t}\cos 2t - \begin{bmatrix}0\\1\end{bmatrix} e^{3t}\sin 2t\\
v_2 &= \Im\left(re^{\lambda_+ t}\right)\\
&= \begin{bmatrix}1\\1\end{bmatrix} e^{3t}\sin 2t + \begin{bmatrix}0\\1\end{bmatrix} e^{3t}\cos 2t.
\end{aligned}$$

Finally, the general solution is then

$$x = c_1 v_1 + c_2 v_2.$$

4 We see that $(0,0)$ is the only equilibrium point of the system.

To find its index for the system, we parametrise a closed curve surrounding it. The simplest such curve is a circle:

$$C : r(t) = (\cos t, \sin t)\,, \quad t \in [0, 2\pi).$$

We then have the following formula for the index:

$$\begin{aligned}
I_{(0,0)} = I_C &= \frac{1}{2\pi}\oint_C d\left(\arctan\frac{f_2(x,y)}{f_1(x,y)}\right)\\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{f_1\dot{f}_2 - f_2\dot{f}_1}{f_1^2 + f_2^2}dt\\
&= \frac{1}{2\pi}\int_0^{2\pi} 2\frac{-\cos^2 2t - \sin^2 2t}{1}dt\\
&= -2\frac{1}{2\pi}\int_0^{2\pi} dt\\
&= -2
\end{aligned}$$

In this computation, we used $f_i(t) = f(r_i(t))$ and the following

$$\begin{aligned}
f_1 &= \cos^2 t - \sin^2 t = \cos 2t,\\
f_2 &= -2\cos t\sin t = -\sin 2t,\\
\dot{f}_1 &= -2\sin 2t,\\
\dot{f}_2 &= -2\cos 2t.
\end{aligned}$$

Note: this can also be done by drawing isoclines and using $I_{(0,0)} = W_{C_f}$.

5 (i) The matrix

$$A = \begin{bmatrix} 0 & -10 & 2\\ 10 & 0 & 1\\ 0 & 0 & -5\end{bmatrix}$$

has eigenvalues $-10i$, $10i$ and -5. Since $\max_i \Re\lambda_i = 0$ and all eigenvalues are distinct, all solutions are stable.

(ii) The matrix

$$A = \begin{bmatrix} -5 & 0 \\ 10 & 0 \end{bmatrix}$$

has eigenvalues -5 and 2. Since $\max_i \Re\lambda_i = 2 > 0$, all solutions are unstable.

(iii) We write the matrix $A = A_0 + B$, with

$$A_0 = \begin{bmatrix} -3 & 2 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix},$$

which has eigenvalues -3 and -7(with multiplicity 2).

Furthermore, $B(t)$ has bounded integral:

$$\int_0^\infty \|B(t)| \, dt \le \int_0^\infty \left(e^{-t} + \frac{1}{1+t^2}\right) dt < \infty.$$

Since $\max_i \Re\lambda_i < 0$ and B has bounded integral, all solutions are asymptotically stable.

$\boxed{6}$ The solution $x(t) = \phi(t; x_0)$ is *stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x_1 - x_0| < \delta \Rightarrow \forall (t > 0) : |x(t) - \phi(t; x_1)| < \varepsilon.$$

Let $x = \phi(t; x_0)$ and $y = \phi(t; y_0)$ be solutions such that $x(0) = x_0$ and $y(0) = y_0$. Then $z = x - y$ solves

$$\dot{z} = Az, \quad z(0) = x_0 - y_0,$$

and

$$z(t) = \Phi(t)\Phi^{-1}(0)z(0).$$

This implies

$$|z(t)| \le \|\Phi(t)\| \cdot \|\Phi^{-1}(0)\| \cdot |x_0 - y_0|.$$

By assumption, $\|\Phi(t)\| \le C$, so

$$|x(t) - y(t)| \le C\|\Phi^{-1}(0)\| \cdot |x_0 - y_0|.$$

This proves that x is stable, and since x was arbitrary, all solutions are stable.

$\boxed{7}$ We try a Liapunov function of the form $V = \frac{1}{2}x^2 + \frac{a}{2}y^2$, and compute its (time) derivative:

$$\begin{aligned} \dot{V} &= x\dot{x} + ay\dot{y} \\ &= 2\left(x^2 + 2y^2\right)xy - x^4 - a\left(x^2 + 2y^2\right)xy - ae^x y^2 \end{aligned}$$

Putting $a = 2$ will rid us of the complicated cross term, leaving us with

$$\dot{V} = -x^4 - 2e^x y^2,$$

which is strictly negative for $(x, y) \neq (0,0)$ and 0 at $(0,0)$.

Furthermore, V is strictly positive for $(x, y) \neq (0,0)$ and 0 at $(0,0)$; and $V \in C^1$. This means that V is a strict Liapunov function for the system, which again implies that $(0,0)$ is an asymptotically stable equilibrium point.

To show that the domain of attraction is all of $\mathbb{R}^2$, we observe that V is a strict Liapunov function in all of $\mathbb{R}^2$, so that by LaSalle's invariance principle, the domain of attraction of $(0,0)$ is all of $\mathbb{R}^2$.

$\boxed{8}$ In this exercise, we choose Liapunov function $V = \frac{1}{2}\left(x^2 + y^2\right)$, with derivative

$$\dot{V} = x\dot{x} + y\dot{y} = \left(x^2 + y^2\right) \sin\left(3x^2 + 2y^2\right)$$

Looking at the sign of sin, we observe the following:

$$\begin{aligned} 0 < 3x^2 + 2y^2 < \pi &\Rightarrow \dot{V} > 0 \\ \pi < 3x^2 + 2y^2 < 2\pi &\Rightarrow \dot{V} < 0 \\ 3x^2 + 2y^2 = n\pi \text{ for } n \in \mathbb{N} &\Rightarrow \dot{V} = 0 \end{aligned}$$

We also have the simple inequality

$$2\left(x^2 + y^2\right) \leq 3x^2 + 2y^2 \leq 3\left(x^2 + y^2\right).$$

The above implies that

$$0 \leq 3\left(x^2 + y^2\right) \leq \pi \Rightarrow \dot{V} \geq 0,$$

for instance: $\dot{V} > 0$ on the circle $x^2 + y^2 = 1 \in (0, \frac{\pi}{3})$; and for

$$3\left(x^2 + y^2\right) \leq 2\pi \text{ and } 2\left(x^2 + y^2\right) \geq \pi \Rightarrow \dot{V} \leq 0,$$

for instance: $\dot{V} \leq 0$ on the circle $x^2 + y^2 = \frac{5\pi}{9} \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$.

Since $\nabla V \neq 0$ for $(x, y) \neq 0$, the domain

$$\Omega := \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, \frac{1}{2} \leq V(x, y) \leq \frac{1}{2}\frac{5}{9}\pi \text{ and } x^2 + y^2 \leq 2\pi \right\}$$

is positively invariant (and closed and bounded).

We check that $(0,0)$ is the only equilibrium point: if $(x, y) \neq (0,0)$ satisfies $f(x, y) = 0$, then $xf_2 - yf_1 = 0$, which is equivalent to $x^2 + y^2 = 0$, contradicting $(x, y) \neq (0,0)$.

Now Poincaré–Bendixson's theorem implies that there exists at least one periodic solution in Ω.

Note: the same argument using polar coordinates is similar in difficulty.

$\boxed{9}$ When $|v| = \frac{1}{\varepsilon}$, $\dot{v} = 0$, so the domain $|v| \leq \frac{1}{\varepsilon}$ is invariant.

Since $|v_0| < \frac{1}{\varepsilon}$, $|v(t)| \leq \frac{1}{\varepsilon}$ for all $t > 0$.

We will show uniqueness of a solution by showing that the difference in norms between any two solutions must be identically zero. So, define $\sigma = |v_1 - v_2|^2$, and differentiate to get

$$\dot{\sigma} = (v_1 - v_2)(\dot{v}_1 - \dot{v}_2) = (v_1 - v_2)(f(v_1) - f(v_2)),$$

where $f(v) = \left(1 - \varepsilon^2 v^2\right)^{\frac{3}{2}}$.

We now wish to find a bound for the second factor above, we can do this using the definition of the derivative:

$$|f(v_1) - f(v_2)| \leq \max_{|v| \leq \frac{1}{\varepsilon}} |f'(v)||v_1 - v_2| \leq 3\varepsilon|v_1 - v_2|,$$

since

$$|f'(v)| = \frac{3}{2}\varepsilon^2|v|\left(1 - \varepsilon^2 v^2\right)^{\frac{1}{2}} \leq 3\varepsilon^2\frac{1}{\varepsilon}1 = 3\varepsilon.$$

This gives

$$\dot{\sigma} \leq 3\varepsilon\sigma,$$

which can be rewritten, using an integrating factor, as

$$\frac{d}{dt}\left(e^{-3\varepsilon t}\sigma\right) \leq 0.$$

Integrating and rearranging leads to

$$\sigma(t) \leq e^{3\varepsilon t}\sigma(0),$$

but $\sigma(0) = 0$, so $\sigma(t)$ is identically 0 for positive t.

This, in turn, implies uniqueness of solutions, which is what we wanted to show.

To see that a solution exists, observe that $f(v)$ is globally Lipschitz continuous for $|v| \leq \frac{1}{\varepsilon}$ and the domain given by $|v| \leq \frac{1}{\varepsilon}$ is invariant, so by the global extension theorem, there is a solution for $|t| \leq T$ for any $T > 0$ implies the result for all $t \in \mathbb{R}$.

Note: Picard–Lindelöf's theorem implies existence for $|t| \leq \min\left(T, \frac{R}{\max|f|}\right) = \min(T, R)$. Since R and T are arbitrary, this means that a solution exists for $t < \infty$.