1a) SYSTEMET ER PÅ FORMEN $\dot{\vec{x}} = A\vec{x}$

$\rightsquigarrow \det(A) \neq \vec{0} \Rightarrow \vec{0}$ ENESTE LIKEVEKTSPUNKT

I VÅRT TILFELLE: $\det(A) = (-4)(-2) + (2-\lambda)\lambda = 8 + 2\lambda - \lambda^2$

$\rightarrow \det(A) = 0 \iff \lambda^2 - 2\lambda - 8 = 0$

$\lambda_{1,2} = 1 \pm \sqrt{1+8} = 1 \pm 3$

$\rightarrow$ MULIGE BIFURKASJONSPUNKTER $\lambda = -2$ OG $\lambda = 4$

($\det(A)$ = PRODUKTET TIL EGENVERDIENE TIL A)

EGENVERDIENE:

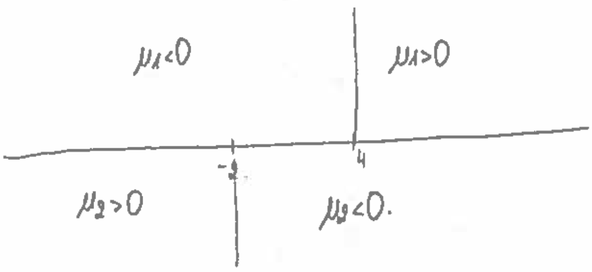
$$\det\begin{pmatrix} -4-\mu & \lambda-2 \\ \lambda & -2-\mu \end{pmatrix} = (4+\mu)(2+\mu) + (2-\lambda)\lambda$$
$$= \mu^2 + 6\mu + 8 + 2\lambda - \lambda^2$$

$$\Rightarrow \mu_{1,2} = -3 \pm \sqrt{9-8-2\lambda+\lambda^2}$$
$$= -3 \pm \sqrt{1-2\lambda+\lambda^2}$$
$$= -3 \pm (1-\lambda)$$

$$\Rightarrow \mu_1 = -3-(1-\lambda) = -4+\lambda$$
$$\mu_2 = -3+(1-\lambda) = -2-\lambda$$

$\Rightarrow \mu_1 < 0$ NÅR $\lambda < 4$ $\qquad \mu_1 > 0$ NÅR $\lambda > 4$

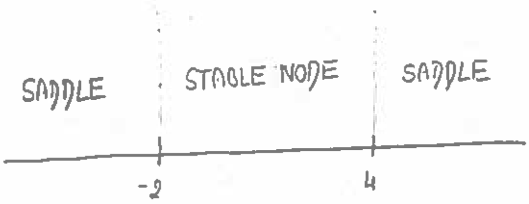
$\mu_2 < 0$ NÅR $-2 < \lambda$ $\qquad \mu_2 > 0$ NÅR $-2 > \lambda$



$\Rightarrow \mu_1 < 0$ & $\mu_2 < 0$ NÅR $-2 < \lambda < 4 \Rightarrow \vec{0}$ ... STABLE NODE

$\mu_1 < 0$ & $\mu_2 > 0$ NÅR $\lambda < -2 \Rightarrow \vec{0}$ ... SADDLE $\rightsquigarrow$ UNSTABLE

$\mu_1 > 0$ & $\mu_2 < 0$ NÅR $\lambda > 4 \Rightarrow \vec{0}$ ... SADDLE $\rightsquigarrow$ UNSTABLE



$\Rightarrow$ TO BIFURKASJONSPUNKTER: $\lambda = -2$ (SADDLE $\rightsquigarrow$ STABLE NODE)

$\lambda = 4$ (STABLE NODE $\rightsquigarrow$ SADDLE)

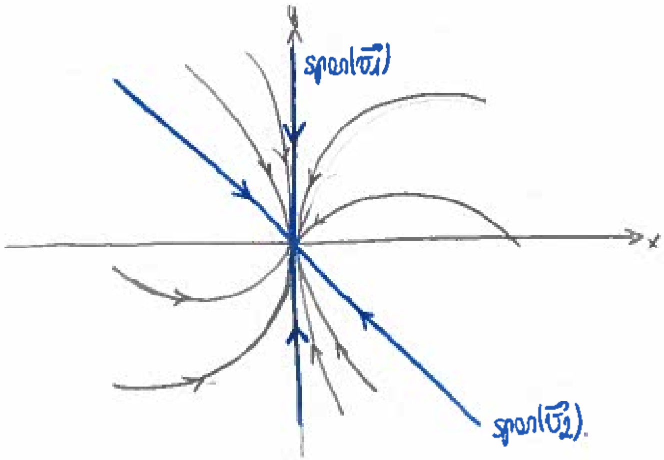
7b.) $\lambda=2$ a) $\Rightarrow$ 0 ENESTE LIKEVEKTSPUNKT
STABLE NODE SIDEN $\mu_1=-2$
$\mu_2=-4$

EGENVEKTORS: $\mu_1=-2$ $\begin{pmatrix}-2 & 0\\ 2 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\vec{0}$

$\Rightarrow$ EGENVEKTOR $\vec{v_1}=\begin{pmatrix}0\\1\end{pmatrix}$

$\mu_2=-4$ $\begin{pmatrix}0 & 0\\ 2 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\vec{0}$

$\Rightarrow$ EGENVEKTOR $\vec{v_2}=\begin{pmatrix}1\\-1\end{pmatrix}$.



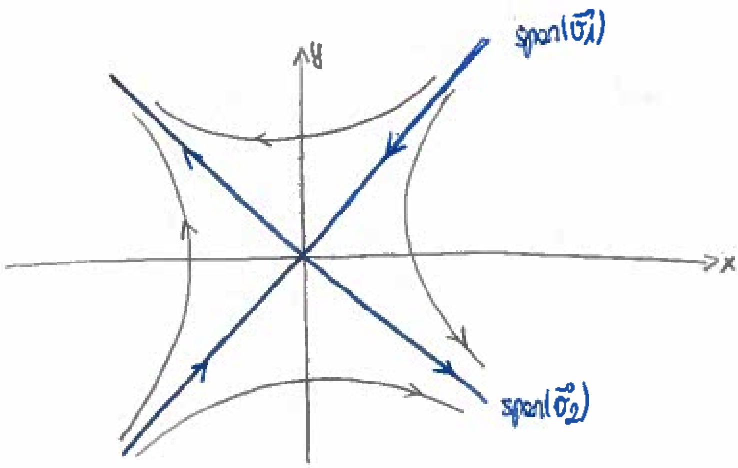
$\lambda=-3$: a) $\Rightarrow$ $\vec{0}$ ENESTE LIKEVEKTSPUNKT
SADDLE SIDEN $\mu_1=-7$
$\mu_2=1$

EGENVEKTORS: $\mu_1=-7$ $\begin{pmatrix}3 & -5\\ -3 & 5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\vec{0}$

$\Rightarrow$ EGENVEKTOR $\vec{v_1}=\begin{pmatrix}5\\3\end{pmatrix}$

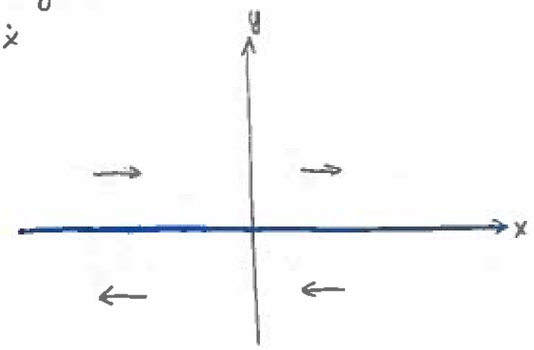
$\mu_2=1$ $\begin{pmatrix}-5 & -5\\ -3 & -3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\vec{0}$

$\Rightarrow$ EGENVEKTOR $\vec{v_2}=\begin{pmatrix}1\\-1\end{pmatrix}$



2a) $\dot{x}=y$

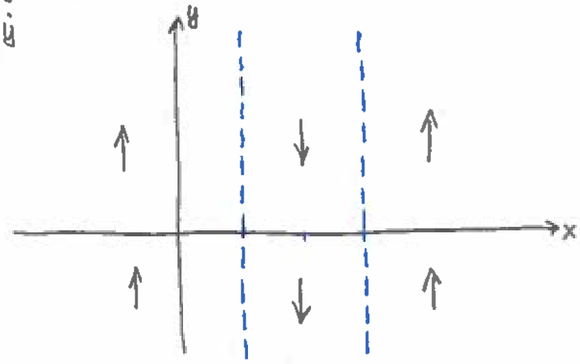
$\dot{x}$



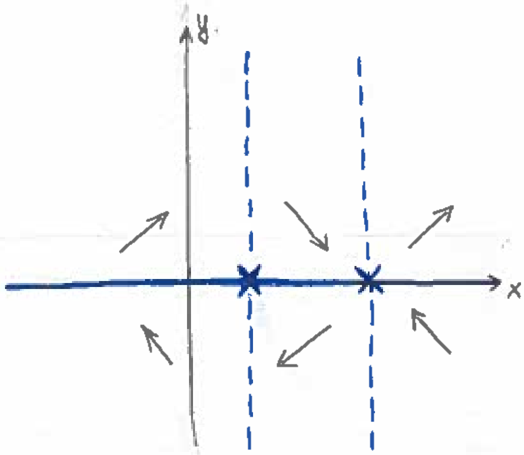
$(— \ldots \dot{x}=0)$.

$\dot{y}=(x^2-4x+3)=(x-3)(x-1)$

$\dot{y}$



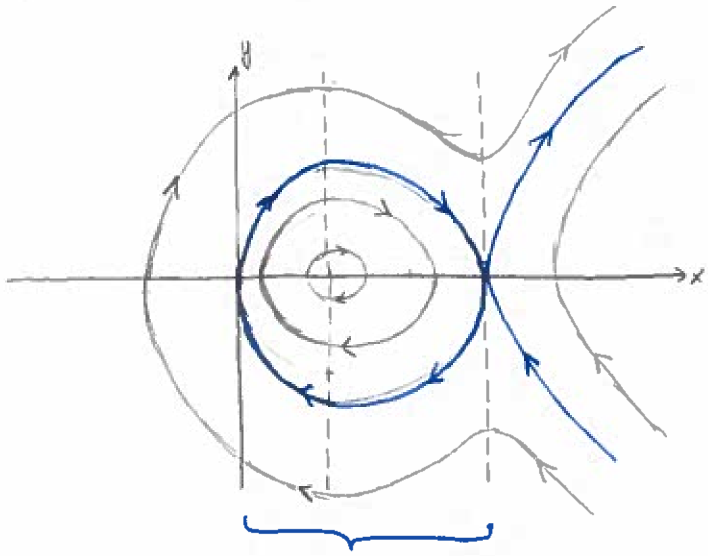
$(- - \ldots \dot{y}=0)$.



TO LIKEVEKTSPUNKTER:

(1,0)... SENTER

(3,0)... SADDLE



HOMOKLIN FASEBANE.

b) HOMOKLIN FASEBANE: FASEBANE SOM FORBINDER EN LIKEVEKTSPUNKT MED SEG SELV.

FASEBANE: REPRESENTERER y SOM EN FUNKSJON AV x:

$$\frac{dy}{dx}=\frac{x^2-4x+3}{y} \Rightarrow y\,dy=(x^2-4x+3)dx$$

$$\Rightarrow \frac{y^2(x)}{2}=\frac{x^3}{3}-2x^2+3x+C$$

(3,0) MÅ LIGGE PÅ FASEBANEN. VI SER ETTER:

$$\Rightarrow 0=y^2(3)=9-18+9+C=C$$

$\Rightarrow$ HOMOKLIN FASEBANE $\{(x,y) \mid \frac{y^2(x)}{2} = \frac{x^3}{3} - 2x^2 + 3x$ OG $x \in [0,3]\}$

(BRUKTE $y^2 = 0 \Leftrightarrow x(\frac{x^2}{3} - 2x + 3) = 0 \Leftrightarrow x(x^2 - 6x + 9) = 0 \Leftrightarrow x(x-3)^2 = 0$)

3). $\dot{x} = -5x + \cos(x) - 1$
$\dot{y} = -2y$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \begin{pmatrix} -5 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cos(x) - 1 \\ 0 \end{pmatrix}$$

1) LINEARISERINGEN I $\vec{0}$: $\begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \begin{pmatrix} -5 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

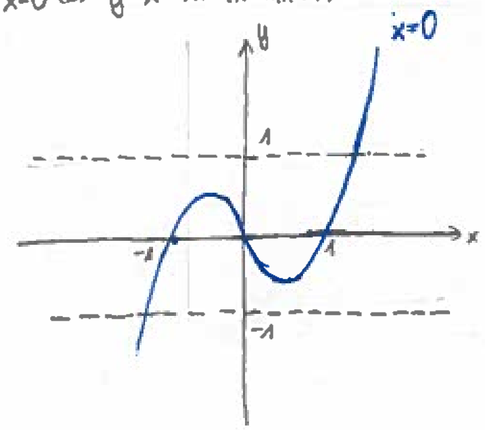
EGENVERDIENE -5 OG -2
$\Rightarrow \vec{0}$ ASYMPTOTISK STABIL

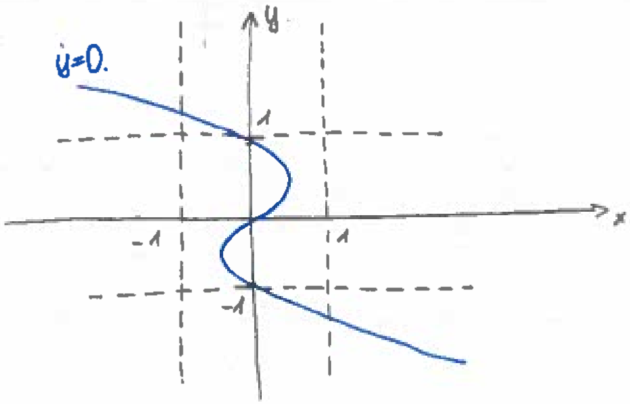
2) $\cos(x) - 1 = \mathcal{O}(x^2)$

$\xRightarrow{1+2}$ $\vec{0}$ ER EN ASYMPTOTISK STABIL LIKEVEKTSPUNKT

4) a) $\dot{x} = x - x^3 + y$
$\dot{x} = 0 \Leftrightarrow y = x^3 - x = (x-1)(x+1)x$

$\dot{y} = y - y^3 - x$
$\dot{y} = 0 \Leftrightarrow x = -y^3 + y = -(y-1)(y+1)y$





$f(x) = x^3 - x \Rightarrow f'(x) = 3x^2 - 1$
$\Rightarrow f'(0) = 0$ NÅR $x = \pm\frac{1}{\sqrt{3}}$
$f(-\frac{1}{\sqrt{3}}) = +\frac{1}{\sqrt{3}}\frac{2}{3} < 1$
$f(\frac{1}{\sqrt{3}}) = -\frac{1}{\sqrt{3}}\frac{2}{3} > -1$

$\Rightarrow$ LIKEVEKTSPUNKT $(0,0)$.

$\dot{x} = x + y - x^3$
$\dot{y} = -x + y - y^3$

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{\text{MATRISE TIL LINEARISERINGEN}} \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} -x^3 \\ -y^3 \end{pmatrix}}_{\mathcal{O}(x^2+y^2)} \Rightarrow (0,0) \text{ USTABILT SPIRAL}$$

$\downarrow$

EGENVERDIER
$1 \pm i$

$\Rightarrow$ USTABILT SPIRAL

4b) · LIAPUNOV FUNKSJON

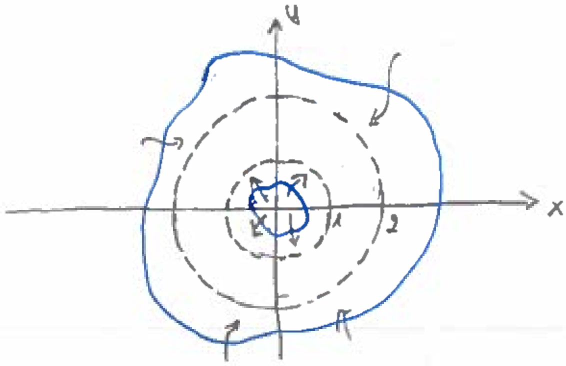
$$V(x,y) = x^2+y^2 \Rightarrow \frac{d}{dt} V(x(t),y(t)) = 2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)$$
$$= 2x^2(t) - 2x^4(t) - 2y^4(t) + 2y^2(t)$$
$$= 2(x^2(t)+y^2(t)) - 2(x^4(t)+y^4(t))$$

$\Rightarrow$ IF $x^2(t)+y^2(t) < 1 \Rightarrow x^4(t)+y^4(t) \leq (x^2(t)+y^2(t))^2 < x^2(t)+y^2(t)$

$$\Rightarrow \frac{d}{dt} V(x(t),y(t)) > 0$$

IF $2 < x^2(t)+y^2(t) \Rightarrow 2(x^2(t)+y^2(t)) < (x^2(t)+y^2(t))^2 = x^4(t) + 2x^2(t)y^2(t) + y^4(t) \leq 2(x^4(t)+y^4(t))$

$$\Rightarrow \frac{d}{dt} V(x(t),y(t)) < 0$$



VELG EN SIMPEL LUKKET KURVE INNENFOR SIRKEL MED RADIUS 1
UTENFOR SIRKEL MED RADIUS 2

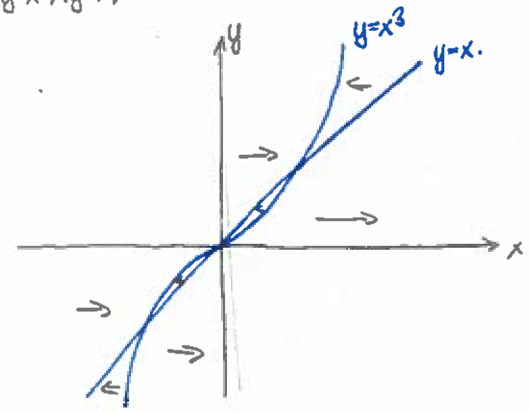
$\Rightarrow$ FORUTSETNINGER TIL POINCARE BENDIXSON OPPFYLLT

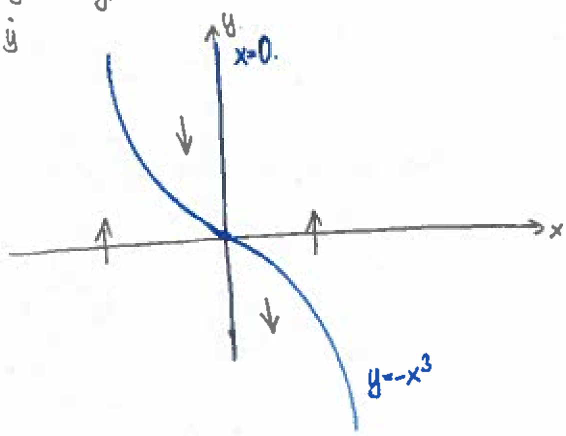
$\Rightarrow$ DET FINNES EN PERIODISK. LØSNING INNENFOR ANNULUSEN BEGRENSET AV DE TO BLÅ KURVENE

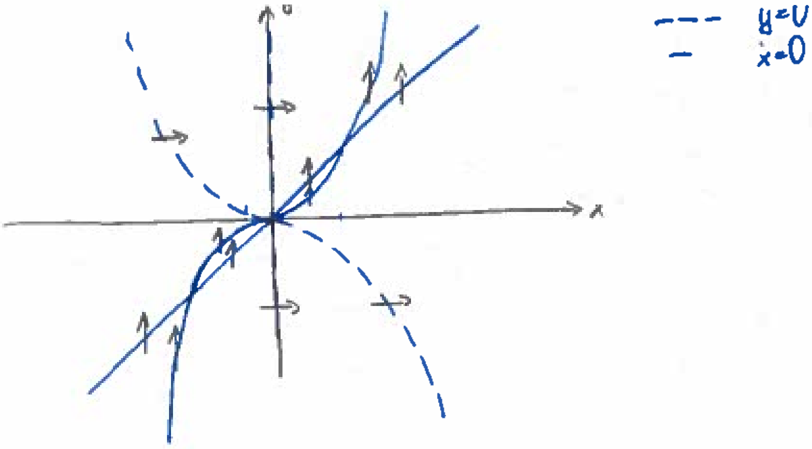
(SIDEN (0,0) ER DEN ENESTE LIKEVEKTSPUNKTEN)

5a) $\dot{x} = (y-x^3)(y-x)$ $\qquad$ $\dot{y} = (x^3+y)x$

$\dot{x}$ $\qquad$ $\dot{y}$







$\Rightarrow$ INDEX=0 $\quad (\uparrow \sim \uparrow \sim \rightarrow \sim \rightarrow \sim \uparrow \sim \uparrow \sim \rightarrow \sim \rightarrow \sim \uparrow)$

5b) TEGNING FRA a) $\Rightarrow$ (0,0) - ENESTE LIKEVEKTSPUNKT
INDEX 0:

$\Rightarrow$ ENHVER PERIODISK LØSNING MÅ GÅ RUNDT $\vec{0}$. OG DERMED HA INDEX 0
↯ (TIL ENHVER PERIODISK LØSNING HAR INDEX 1)

$\Rightarrow$ DET FINNES INGEN IKKE-KONSTANTE PERIODISKE LØSNINGER

6)
$$\dot{x} = x^2(1 + x + x^3 + x^{100})$$
$$x(0) = x_0 > 0$$

HØYRE SIDE ·) LIPSCHITZ KONTINUERLIG $\Rightarrow$ LOKAL EKSISTENS OG ENTYDIGHET AV LØSNINGER
·) POSITIVE NÅR $x > 0$ $\Rightarrow$ $x(t) > 0$ $\forall t$ NÅR $x(0) > 0$

$$\dot{x} = x^2(1 + x + x^3 + x^{100}) \geq x^2$$

SER PÅ $\dot{x} \geq x^2 \Rightarrow \frac{\dot{x}}{x^2} \geq 1 \quad \Big| \int_0^t$

$$\Rightarrow t \leq \int_0^t \frac{\dot{x}(s)}{x^2(s)} ds = \int_{x(0)}^{x(t)} \frac{1}{y^2} dy = -\frac{1}{y}\Big|_{x(0)}^{x(t)} = -\frac{1}{x(t)} + \frac{1}{x(0)} = \frac{x(t) - x(0)}{x(0)x(t)}$$

$$\Rightarrow x(0)x(t)t \leq x(t) - x(0)$$

$$\Rightarrow x(0) \leq (1 - x(0)t)\, x(t)$$

$$\Rightarrow \underbrace{\frac{x(0)}{1 - x(0)t}}_{\to\, -\infty \ (t \to \frac{1}{x(0)})} \leq x(t)$$

$$\Rightarrow \exists t^* < \frac{1}{x(0)} \text{ SÅ } \lim_{t \uparrow t^*} x(t) = \infty$$