Problem 1 Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

a) Sketch the phase diagram of the system with orientations.

b) Compute the solution to the initial value problem

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Problem 2 Consider the system

$$\begin{aligned} \dot{x} &= y(1-y) \\ \dot{y} &= x. \end{aligned}$$

a) Sketch the phase diagram of the system with orientations.

b) Determine whether or not the system has non-constant periodic solutions. Determine whether or not the system has solutions $(x(t), y(t))$ which satisfy

$$\lim_{t\to\pm\infty} x(t) = 0 \quad \text{and} \quad \lim_{t\to\pm\infty} y(t) = 1.$$

Hint: How are such solutions represented in the phase diagram?

Problem 3 Consider the system

$$\begin{aligned} \dot{x} &= xg(x,y) + y \\ \dot{y} &= -x + yg(x,y). \end{aligned}$$

where $g(x, y) = 3 + 2x - x^2 - y^2$.

a) Determine if the origin is a stable, asymptotically stable or unstable equilibrium point for the system.

b) Determine whether or not this system has non-constant periodic solutions.

Problem 4 Consider the system

$$\dot{x} = (1 + e^{-t})x + \frac{1 - 2e^t}{1 + e^t}y$$
$$\dot{y} = 3x - \frac{4t^2 + 7}{1 + t^2}y.$$

Determine if the origin is a stable, asymptotically stable or unstable equilibrium point for the system.

Problem 5 Consider the initial value problem

$$\dot{x} = -t^3 x, \quad x(0) = x_0 \in \mathbb{R}.$$

Define what it means for a solution to be asymptotically stable.
Show that all solutions to the above differential equation are asymptotically stable.

Problem 6 Consider the system

$$\dot{x} = (y - x)(y + x)y$$
$$\dot{y} = (y - x^2)(y + x^2)$$

Compute the index of the origin.

Problem 7 Consider the initial value problem

$$\dot{x} = 2 - \cos^2(x), \quad x(0) = x_0 \in \mathbb{R}.$$

Show, with the help of a Grönwall estimate, that the initial value problem cannot have more than one solution.