1) a) $\dot{\vec{x}} = A\vec{x}$ WITH $A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$

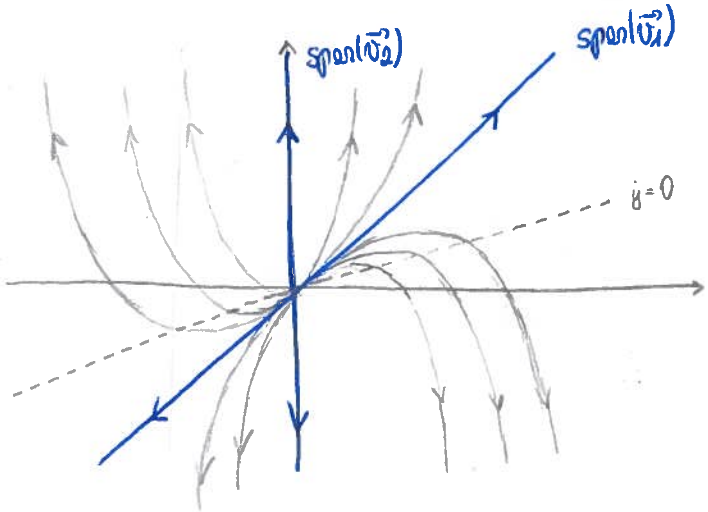
·) EIGENVALUES OF A: $\lambda = 1$, $\lambda = 2$ (SINCE A IS LOWER TRIANGULAR

$\Rightarrow$ EIGENVALUES $\hat{=}$ ELEMENTS ON THE DIAGONAL)

$\Rightarrow$ UNSTABLE NODE.

·) EIGENVECTORS: $\vec{v}_1$ $\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $(\sim \lambda = 1!)$

$\vec{v}_2$: $\begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \vec{v}_2 = 0 \Rightarrow \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $(\sim \lambda = 2!)$



b) THE INITIAL VALUE PROBLEM IS OF THE FORM

$\dot{\vec{x}} = A\vec{x} + b$, $\vec{x}(0) = \vec{x}_0$

$\Rightarrow$ BY THE VARIATION OF CONSTANT FORMULA:

$$\vec{x}(t) = \Phi(t)\Phi^{-1}(0)\vec{x}_0 + \int_0^t \Phi(t)\Phi^{-1}(s)\, b\, ds$$

WHERE $\Phi(t)$ DENOTES A FUNDAMENTAL MATRIX TO $\dot{\vec{x}} = A\vec{x}$

·) FUNDAMENTAL MATRIX TO $\dot{\vec{x}} = A\vec{x}$

$1, 2$ EIGENVALUES TO A

$\Rightarrow$ $x(t) = ae^{t} + be^{2t}$

$y(t) = ce^{t} + de^{2t}$

$\Rightarrow$ $\dot{x}(t) = ae^{t} + 2be^{2t} = ae^{t} = x(t)$

$\dot{y}(t) = ce^{t} + 2de^{2t} = -ae^{t} - be^{2t} + 2ce^{t} + 2de^{2t} = -x(t) + 2y(t)$

$\Rightarrow$ $b = 0$, $c = -a + 2c$ $\Rightarrow$ $b = 0$

$2d = -b + 2d$ $\quad$ $c = a$

$a, d \in \mathbb{R}$

$$\Rightarrow \begin{array}{l} x(t) = a e^t \\ y(t) = a e^t + d e^{2t} \end{array} \quad \text{OR} \quad \vec{x}(t) = \underbrace{\begin{pmatrix} e^t & 0 \\ e^t & e^{2t} \end{pmatrix}}_{\Phi(t)} \begin{pmatrix} a \\ d \end{pmatrix}$$

$$\Phi(t) = \begin{pmatrix} e^t & 0 \\ e^t & e^{2t} \end{pmatrix} \Rightarrow \Phi^{-1}(t) = e^{-3t} \begin{pmatrix} e^{2t} & 0 \\ -e^t & e^t \end{pmatrix}$$

$$\Phi(0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Rightarrow \Phi^{-1}(0) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

·) $\dot{\vec{x}} = A\vec{x} + b$, $\vec{x}(0) = \vec{x}_0$ WITH $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \vec{x}(t) = \begin{pmatrix} e^t & 0 \\ e^t & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} e^t & 0 \\ e^t & e^{2t} \end{pmatrix} \begin{pmatrix} e^{2s} & 0 \\ -e^s & e^s \end{pmatrix} \begin{pmatrix} e^{-3s} \\ 0 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^t & 0 \\ e^t & e^{2t} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \int_0^t \begin{pmatrix} e^t & 0 \\ e^t & e^{2t} \end{pmatrix} \begin{pmatrix} e^{-s} \\ -e^{-2s} \end{pmatrix} ds$$

$$= \begin{pmatrix} e^t \\ e^t - e^{2t} \end{pmatrix} + \int_0^t \begin{pmatrix} e^{t-s} \\ e^{t-s} - e^{2t-2s} \end{pmatrix} ds$$

$$= \begin{pmatrix} e^t \\ e^t - e^{2t} \end{pmatrix} + \begin{pmatrix} -e^{t-s} \Big|_0^t \\ -e^{t-s} + \frac{1}{2} e^{2t-2s} \Big|_0^t \end{pmatrix}$$

$$= \begin{pmatrix} e^t \\ e^t - e^{2t} \end{pmatrix} + \begin{pmatrix} -1 + e^t \\ -1 + \frac{1}{2} + e^t - \frac{1}{2} e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2e^t - 1 \\ 2e^t - \frac{3}{2} e^{2t} - \frac{1}{2} \end{pmatrix} \Rightarrow \begin{array}{l} x(t) = 2e^t - 1 \\ y(t) = 2e^t - \frac{3}{2} e^{2t} - \frac{1}{2} \end{array}$$

⌈CHECK $\vec{x}(0) = 1$ ✓ $\quad \dot{x}(t) = 2e^t = 2e^t - 1 + 1 = x(t) + 1$ ✓

$\vec{y}(0) = 0$ ✓ $\quad \dot{y}(t) = 2e^t - 3e^{2t}$

$-x(t) + 2y(t) = -2e^t + \cancel{1} + 4e^t - 3e^{2t} - \cancel{1} = 2e^t - 3e^{2t}$

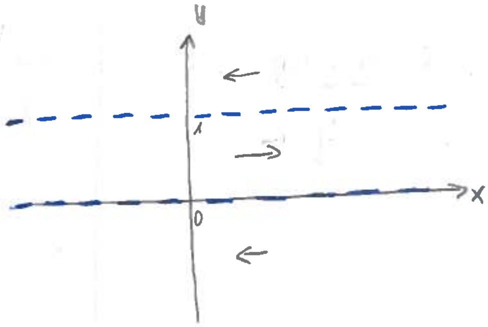
$\Rightarrow \dot{y}(t) = -x(t) + 2y(t)$ ✓ ⌋

2a) $\dot{x} = y(1-y)$
$\dot{y} = x$

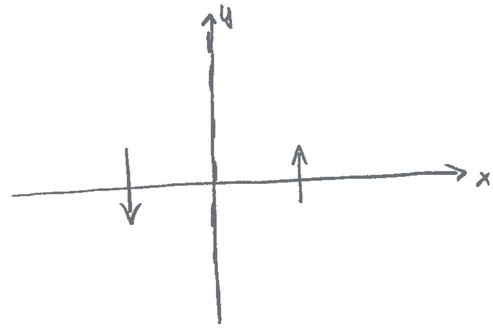
·) EQUILIBRIUM POINTS:

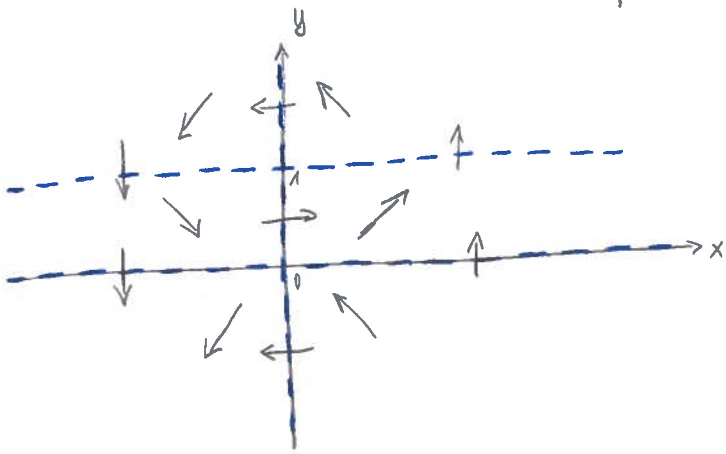
$$\left.\begin{array}{l}\dot{x}=0 \Rightarrow y=0 \vee y=1 \\ \dot{y}=0 \Rightarrow x=0\end{array}\right\} \Rightarrow (0,0),\ (0,1)$$

·) $\dot{x}$



$\dot{y}$





SEEMS LIKE (0,0) IS A SADDLE
(0,1) IS A CENTRE

·) (0,0): LINEARIZATION: $\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x}$

(EIGENVALUES: $\lambda^2 - 1 = 0 \Rightarrow \lambda_{1,2} = \pm 1$
EIGENVECTORS: $\vec{v}_1$: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \vec{v}_1 = \vec{0}$ $\quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{v}_{-1}$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{v}_{-1} = \vec{0}$ $\quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$)

·) (0,1): LINEARIZATION: $\dot{\vec{x}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}$

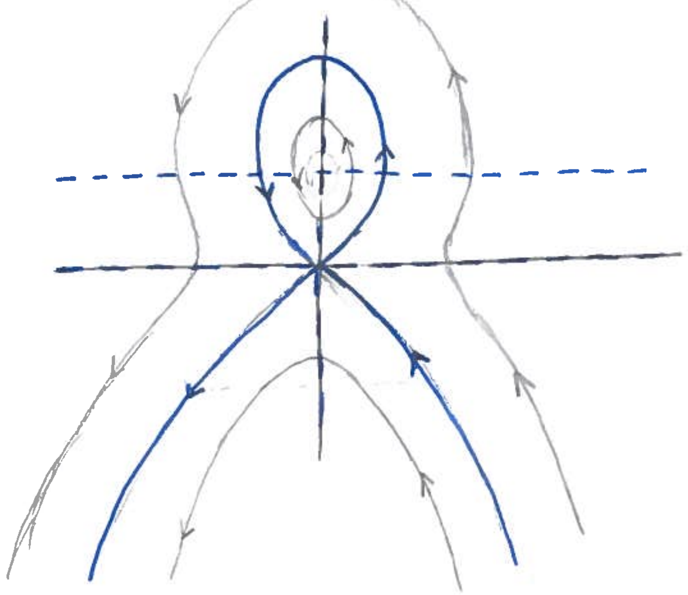
EIGENVALUES: $\lambda_{1,2} = \pm i \Rightarrow$ LINEARIZATION OF NO HELP
(UNLESS YOU SHOW THAT THE SYSTEM IS HAMILTONIAN)

OTHER ARGUMENT:

PHASE PATHS: $\frac{dy}{dx} = \frac{x}{y(1-y)} \Rightarrow x\,dx = y(1-y)\,dy$

$\Rightarrow \frac{x^2}{2} = \frac{y^2}{2} - \frac{y^3}{3} + C \quad$ C.. CONSTANT

$\Rightarrow$ SYMMETRIC WRT THE x-AXIS.



b) NON-CONSTANT PERIODIC SOLUTIONS $\triangleq$ CLOSED PHASE PATHS.

SINCE THERE EXIST CLOSED PHASE PATHS SURROUNDING $(0,1)$

THE ANSWER IS YES.

SOLUTIONS $(x(t), y(t))$ SATISFYING: $\lim_{t\to\pm\infty} x(t)=0$, $\lim_{t\to\pm\infty} y(t)=1$ $\triangleq$ HOMOCLINIC PHASE PATHS CONNECTING $(0,1)$ WITH ITSELF OR THE CONSTANT SOLUTION $(x(t),y(t))=(0,1)$

·) ONLY HOMOCLINIC PHASE PATH: THE PART OF THE SEPARATRIX IN THE UPPER HALF-PLANE

$\Rightarrow$ CONNECTS $(0,0)$ WITH ITSELF

·) $(0,1)$: EQUILIBRIUM POINT

$\Rightarrow (x(t),y(t))=(0,1)\ \forall t$ IS A SOLUTION $\Rightarrow$ THE ANSWER IS YES.

3) a.)
$$\dot{x}= x g(x,y)+y = 3x+2x^2-x^3-xy^2+y \qquad (1)$$
$$\dot{y}=-x+3y+2xy-x^2y-y^3$$

$\Rightarrow$ LINEARIZATION AROUND $(0,0)$ IS GIVEN BY $\dot{\vec{x}}=\begin{pmatrix}3 & 1\\ -1 & 3\end{pmatrix}\vec{x}=A\vec{x}$

SINCE:
$$\left|\frac{2x^2-x^3-xy^2}{\sqrt{x^2+y^2}}\right| \leq 2\,\frac{x^2+|x^3|}{|x|}+\frac{|x|y^2}{|y|}=2(|x|+x^2)+|x|y\to 0 \text{ AS } (x,y)\to 0$$
$$\left|\frac{2xy-x^2y-y^3}{\sqrt{x^2+y^2}}\right| \leq \frac{2|x||y|+|x^2||y|+|y|^3}{|y|}=2|x|+x^2+y^2\to 0 \text{ AS } (x,y)\to 0$$

A: HAS EIGENVALUES $\lambda_{1,2}=3\pm i$ $\rightarrow$ UNSTABLE SPIRAL

$\Rightarrow$ THE ORIGIN OF BOTH THE LINEARIZATION & (1) IS UNSTABLE.

b) $\dot{x} = x g(x,y) + y$
$\dot{y} = -x + y g(x,y)$

EQUILIBRIUM POINTS:

$$\left.\begin{matrix}\dot{x}=0\\ \dot{y}=0\end{matrix}\right\} \Rightarrow \left.\begin{matrix} y = -x g(x,y) \\ x = y g(x,y)\end{matrix}\right\} \Rightarrow y = -y g(x,y)^2 \Rightarrow y(1+g(x,y)^2) = 0 \Rightarrow y=0 \Rightarrow x=0$$

$\Rightarrow (0,0)$ IS THE ONLY EQUILIBRIUM POINT.

FROM a) UNSTABLE SPIRAL

$\Rightarrow I_{(0,0)} = 1 \Rightarrow$ THERE COULD BE PERIODIC SOLUTIONS SURROUNDING $(0,0)$.

FIND A SUITABLE LIAPUNOV FCT, SO THAT ONE COULD USE POINCARÉ BENDIXSON:

$$V(x(t),y(t))^{\cdot} = V_x \dot{x}(t) + V_y \dot{y}(t)$$
$$= V_x x g(x,y) + V_x y - V_y x + V_y y g(x,y)$$
$$\overset{!}{=} (x^2+y^2) g(x,y)$$

IF WE CHOOSE $V_x = x$, $V_y = y$ $\rightsquigarrow$ $V(x,y) = \frac{1}{2}(x^2+y^2)$

$\Rightarrow$ SIGN OF $\dot{V}(x(t),y(t))$ DEPENDENT ON THE SIGN OF $g(x,y)$

$$g(x,y) = 3 + 2x - x^2 - y^2 = 4 - (x-1)^2 - y^2$$

$\Rightarrow g(x,y) = 0$ ON THE CIRCLE WITH RADIUS 2 CENTERED AT $(1,0)$. $(\triangleq C_g)$

$g(x,y) > 0$ INSIDE THE CIRCLE WITH RADIUS 2 CENTERED AT $(1,0)$

$g(x,y) < 0$ OUTSIDE THE CIRCLE WITH RADIUS 2 CENTERED AT $(1,0)$.

$\Rightarrow$ FIND A CIRCLE C_r CENTERED AT $(0,0)$ WHICH LIES INSIDE C_g.

$(x,y) \in C_r \Rightarrow x^2+y^2 = r^2$

$$\Rightarrow g(x,y) = 3 + 2x - x^2 - y^2 = 3 + 2x - r^2 \geq 3 - 2r - r^2 > 0$$

$$r^2 + 2r - 3 = 0 \Rightarrow r = -1 \pm \sqrt{1+3} = -1 \pm 2$$

$\Rightarrow r < 1.$

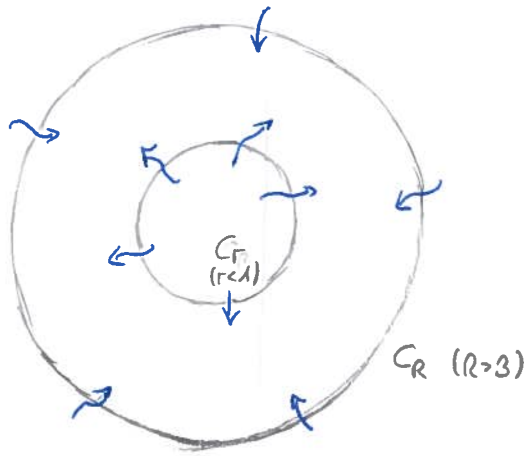
$\Rightarrow$ FIND A CIRCLE C_R CENTERED AT $(0,0)$ WHICH LIES OUTSIDE C_g

$(x,y) \in C_R \Rightarrow x^2+y^2 = R^2$

$$\Rightarrow g(x,y) = 3 + 2x - x^2 - y^2 = 3 + 2x - R^2 \leq 3 + 2R - R^2 < 0$$

$$R^2 - 2R - 3 = 0 \Rightarrow R_{1,2} = 1 \pm \sqrt{1+3} = 1 \pm 2.$$

$\Rightarrow R > 3.$



LET $\{(x,y): r^2 \leq x^2+y^2 \leq R^2\} = \mathcal{J}$ $(r<1<3<R)$

THEN POINCARÉ-BENDIXSON IMPLIES THE EXISTENCE OF A NON-CONSTANT PERIODIC SOLUTION IN $\mathcal{J}$ (SINCE $\mathcal{J}$ CONTAINS NO EQUILIBRIUM POINTS.

4) $\dot{x} = (1+e^{-t})x + \frac{1-2e^t}{1+e^t}y = x - 2y + e^{-t}x + \frac{3}{1+e^t}y$

$\dot{y} = 3x - \frac{4t^2+7}{1+t^2}y = 3x - 4y - \frac{3}{1+t^2}y$

$\Rightarrow$ THE SYSTEM CAN BE WRITTEN AS:

$$\dot{\vec{x}} = A\vec{x} + C(t)\vec{x} \quad \text{WHERE} \quad A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \quad C(t) = \begin{pmatrix} e^{-t} & \frac{3}{1+e^t} \\ 0 & -\frac{3}{1+t^2} \end{pmatrix} \qquad (2)$$

EIGENVALUES OF A: $(1-\lambda)(-4-\lambda)+6 = -4+4\lambda-\lambda+\lambda^2+6 = \lambda^2+3\lambda+2$

$\rightarrow \lambda_{1,2} = -\frac{3}{2} \pm \sqrt{\frac{9}{4}-2} = -\frac{3}{2} \pm \frac{1}{2} = \begin{matrix} -2 \\ -1 \end{matrix}$

$\rightarrow$ STABLE NODE

$\rightarrow$ STABLE & ASYMPTOTIC STABLE FOR $\dot{\vec{x}} = A\vec{x}$

ALSO TRUE FOR (2) IF $\int_0^t \|C(s)\| ds < \infty \quad \forall t \geq 0$

·) $\int_0^t \frac{1}{1+e^s} ds \leq \int_0^t \frac{1}{e^s} ds = \int_0^t e^{-s} ds = -e^{-s}\Big|_0^t = 1 - e^{-t} \leq 1$ ✓

·) $\int_0^t \frac{1}{1+s^2} ds \leq \int_0^1 1 ds + \int_1^t \frac{1}{s^2} ds = 1 - \frac{1}{s}\Big|_1^t = 1 - \frac{1}{t} + 1 \leq 2$ IF $t \geq 1$

$\int_0^t \frac{1}{1+s^2} ds \leq t$ IF $t \leq 1$

$\Rightarrow$ (0,0) IS ASYMPTOTICALLY STABLE AND HENCE ALSO STABLE

5) A SOLUTION $x^*(t)$ IS ASYMPTOTICALLY STABLE IF THERE EXISTS $\eta>0$ ST

$$\|x^*(t_0)-x(t_0)\|<\eta \Rightarrow \lim_{t\to\infty}\|x^*(t)-x(t)\|=0.$$

$\dot{x}(t)=-t^3x(t) \Rightarrow$

SEEMS LIKE ALL SOLUTIONS TEND TO 0 AS $t\to\infty$.

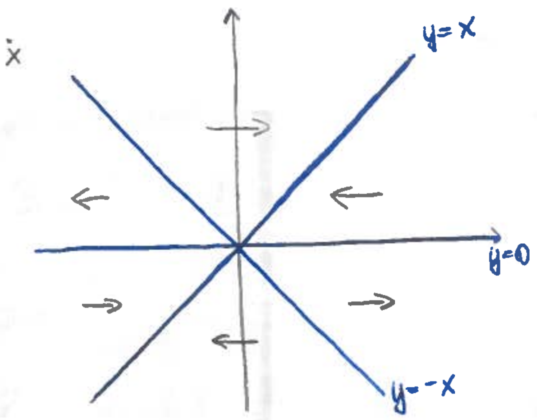
$$\frac{\dot{x}(t)}{x(t)}=-t^3 \Big| \int_0^t \Rightarrow x(t)=x(0)e^{-\frac{t^4}{4}} \Rightarrow |x(t)|\le|x(0)|e^{-t^4/4}\to 0 \text{ AS } t\to\infty$$

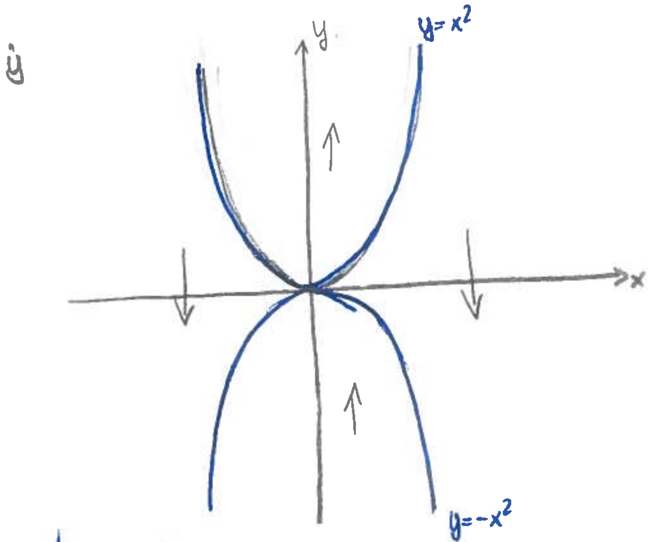
$$\Rightarrow |x^*(0)-x(0)|<\eta$$

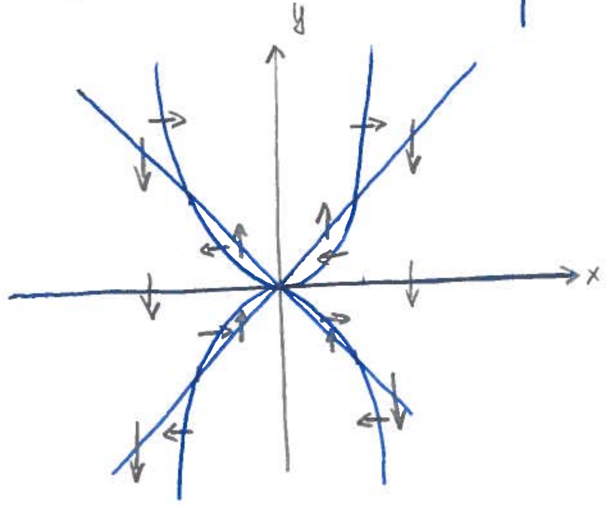
$$\Rightarrow |x^*(t)-x(t)|=|x(0)-x^*(0)|e^{-t^4/4}<\eta e^{-t^4/4}\to 0 \text{ AS } t\to\infty$$

6) $\dot{x}=(y-x)(y+x)y$
$\dot{y}=(y-x^2)(y+x^2)$

(0,0) EQUILIBRIUM POINT, BUT MAYBE NOT THE ONLY ONE.

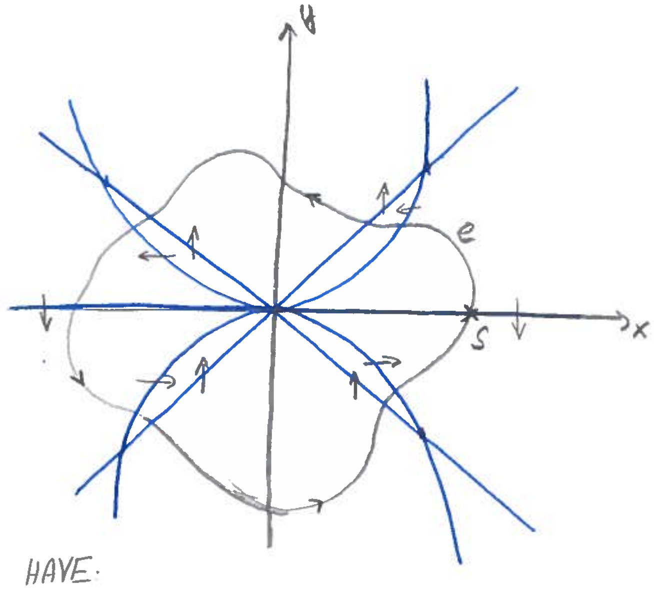






$\Rightarrow$ 5 EQUILIBRIUM POINTS. (0,0)
(1,1), (-1,1), (1,-1), (-1,-1)

HAVE TO CHOOSE A COUNTER CLOCKWISE CURVE WHICH ONLY SURROUNDS (0,0)
BUT NO OTHER EQUILIBRIUM POINT



STARTING AT S: WE HAVE:

$\downarrow \leadsto \leftarrow \leadsto \uparrow \sim \uparrow \leadsto \leftarrow \leadsto \downarrow \leadsto \rightarrow \leadsto \uparrow \sim \uparrow \leadsto \rightarrow \leadsto \downarrow$

$\Rightarrow I_{(0,0)} = 0$

4) ASSUME THAT THERE EXIST TWO SOLUTIONS $x(t) \neq y(t)$

$\Rightarrow z(t) = x(t) - y(t)$ SATISFIES

$$\frac{d}{dt} z^2(t) = 2z(t)\,\dot{z}(t) = 2z(t)\,(\dot{x}(t) - \dot{y}(t))$$
$$= 2z(t)\left(-\cos^2(x(t)) + \cos^2(y(t))\right)$$
$$\leq 4z(t)^2$$

WHERE WE USED: $\cos^2(y(t)) - \cos^2(x(t)) = \int_{x(t)}^{y(t)} -2\cos(r)\sin(r)\,dr$

$$\Rightarrow \left|\cos^2(y(t)) - \cos^2(x(t))\right| \leq 2\,|y(t) - x(t)| = 2|z(t)|$$

$\Rightarrow \frac{d}{dt} z^2(t) \leq 4z(t)^2 \Rightarrow z^2(t) \leq z^2(0)e^{4t}$

AND HENCE $z^2(0) = 0 \Rightarrow z(t)^2 = 0 \ \forall t$

$\Rightarrow x(t) = y(t) \ \forall t$ $\lightning$ (TO $x(t) \neq y(t)$)

$\Rightarrow$ THE INITIAL VALUE PROBLEM CANNOT HAVE MORE THAN ONE SOLUTION!