1) LET $\vec{x}=\begin{pmatrix}x\\y\end{pmatrix}$ THEN $\dot{\vec{x}}=A\vec{x}$ WHERE $A=\begin{pmatrix}-5 & -2\\ 4 & 1\end{pmatrix}$

- $\det(A)=(-5)\cdot 1-(-2)\cdot 4=-5+8=3\neq 0$

  $\Rightarrow \vec{x}=\vec{0}$ IS THE ONLY EP.

- EIGENVALUES & EIGENVECTORS OF A:

$$\det(A-\lambda I)=(-5-\lambda)(1-\lambda)+8=-5-\lambda+5\lambda+\lambda^2+8$$
$$=\lambda^2+4\lambda+3=0$$
$$\Rightarrow \lambda_{1,2}=-2\pm\sqrt{4-3}=-2\pm 1 \;\begin{matrix}\nearrow -1\\ \searrow -3.\end{matrix}$$

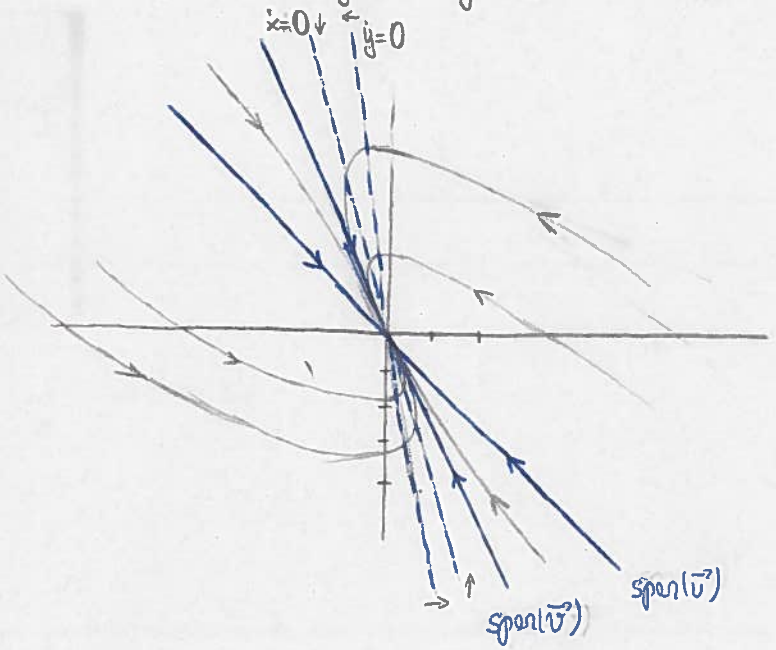
$\Rightarrow$ STABLE NODE.

$\lambda_1=-1$: $\quad (A+I)\vec{v}=0: \Rightarrow -4v_1-2v_2=0 \Rightarrow \vec{v}=\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}1\\-2\end{pmatrix}$

$\lambda_2=-3$ $\quad (A+3I)\vec{w}=0 \Rightarrow -2w_1-2w_2=0 \Rightarrow \vec{w}=\begin{pmatrix}w_1\\w_2\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$

- ISOCLINES: $\quad \dot{x}=0 \Rightarrow -5x-2y=0 \Rightarrow y=-\frac{5}{2}x \quad \rightsquigarrow \dot{y}=4x+y=\frac{3}{2}x$

  $\dot{y}=0 \Rightarrow 4x+y=0 \Rightarrow y=-4x \quad \rightsquigarrow \dot{x}=-5x-2y=3x$



2) $\begin{pmatrix}x\\y\end{pmatrix}^{\cdot}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=A\begin{pmatrix}x\\y\end{pmatrix} \quad \Rightarrow \det(A)=ad-bc\neq 0 \Rightarrow \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ IS THE ONLY EP.

NOTE IF THE PHASE DIAGRAM WITH ORIENTATIONS IS SYMMETRIC WRT THE y-AXIS THEN

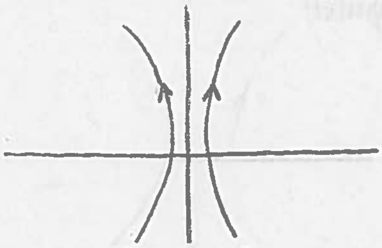
- NO PHASE PATH CAN CROSS THE y-AXIS, BUT
- $\{(0,y): y>0\}$ IS A PHASE PATH
- $\{(0,y): y<0\}$ IS A PHASE PATH
- $\{(0,0)\}$ IS A PHASE PATH

ALTERNATIVE 1:

LET $(x(t), y(t))$ BE A SOL TO THE GIVEN DYNAMICAL SYSTEM

$\Rightarrow$ $\{(x(t), y(t))\ t \in \mathbb{R}\}$ ITS PHASE PATH.

$\Rightarrow \exists g(t)$ WITH $\dot g(t) > 0\ \forall t$ ST $(\tilde x(t), \tilde y(t)) = (-x(g(t)), y(g(t)))$ IS A SOLUTION TO THE GIVEN DYNAMICAL SYSTEM & ITS PHASE PATH $\{(\tilde x(t), \tilde y(t)) : t \in \mathbb{R}\}$ IS SYMMETRIC WRT THE y-AXIS TO THE ONE OF $(x(t), y(t))$



SOL $(\tilde x(t), \tilde y(t))$ SOL $(x(t), y(t))$

$$\dot{\tilde x}(t) = -(x(g(t)))^{\cdot} = -\dot x(g(t))\dot g(t) = -\dot g(t)\left(ax(g(t)) + by(g(t))\right) = \dot g(t)\left(a\tilde x(t) - b\tilde y(t)\right) \overset{?}{=} a\tilde x(t) + b\tilde y(t)$$

$$\dot{\tilde y}(t) = (y(g(t)))^{\cdot} = \dot y(g(t))\dot g(t) = \dot g(t)\left(cx(g(t)) + dy(g(t))\right) = \dot g(t)\left(-c\tilde x(t) + d\tilde y(t)\right) \overset{?}{=} c\tilde x(t) + d\tilde y(t)$$

$\Rightarrow$ HAVE TO FIND OUT UNDER WHICH ASSUMPTIONS WE HAVE

$$\dot g(t)\left(a\tilde x(t) - b\tilde y(t)\right) = a\tilde x(t) + b\tilde y(t)$$
$$\dot g(t)\left(-c\tilde x(t) + d\tilde y(t)\right) = c\tilde x(t) + d\tilde y(t)$$

NOTE $g(t)$ DEP ON THE PHASE PATH & THEREFORE IMPLICITELY ON $(\tilde x(t), \tilde y(t)) \in \mathbb{R}^2$ BUT FOR EACH POINT $(\tilde x(t), \tilde y(t)) \in \mathbb{R}^2$ $\dot g(t) > 0$!

IF $(\tilde x(t), \tilde y(t)) = (0, 1)$ $\Rightarrow -\dot g(t) b = b \Rightarrow b = 0$

IF $(\tilde x(t), \tilde y(t)) = (1, 0)$ $\Rightarrow -\dot g(t) c = c \Rightarrow c = 0$

$\Rightarrow$ FOR ARBITRARY $(\tilde x(t), \tilde y(t))$: $a\dot g(t)\tilde x(t) = a\tilde x(t) \Rightarrow \dot g(t) = 1$

$d\dot g(t)\tilde y(t) = d\tilde y(t) \Rightarrow \dot g(t) = 1$

$\Rightarrow b = c = 0$ & $a, d \in \mathbb{R}\setminus\{0\}$.

ALTERNATIVE 2:

FOR EACH PHASE PATH WE HAVE $\left.\dfrac{dy}{dx}\right|_{(x,y)} = \dfrac{cx + dy}{ax + by}$

IF THE PHASE DIAGRAM WITH ORIENTATIONS IS SYMMETRIC WRT THE y-AXIS

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(x,y)} = -\left.\frac{dy}{dx}\right|_{(-x,y)}$$ (CONTAINS NO INFORMATION ABOUT THE ORIENTATION)

NOTE: $\frac{dy}{dx}$ ONLY MAKES SENSE IF $\dot{x} \neq 0$

$\dot{x} = 0$ HOLDS FOR ALL POINTS $(x,y) = (0,y)$ $(y \in \mathbb{R})$

$\Rightarrow b = 0$ & $a \neq 0$.

$\Rightarrow \forall (x,y)$ ST $\dot{x} \neq 0$

$$\frac{cx+dy}{ax} = \left.\frac{dy}{dx}\right|_{(x,y)} = -\left.\frac{dy}{dx}\right|_{(-x,y)} = -\frac{-cx+dy}{-ax} = \frac{-cx+dy}{ax}$$

$\Rightarrow cx + dy = -cx + dy$

$\Rightarrow c = 0$ & $d \neq 0$

$\Rightarrow b = c = 0$ & $a, d \in \mathbb{R}\setminus\{0\}$ AND $\begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

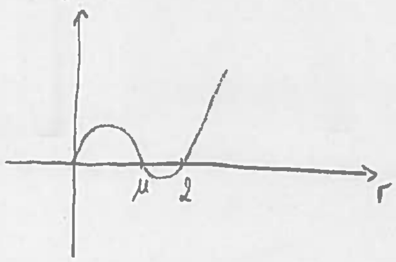
DIRECT COMPUTATIONS YIELD IF $(x(t), y(t))$ IS A SOLUTION, THEN ALSO $(\tilde{x}(t), \tilde{y}(t)) = (-x(t), y(t))$ IS A SOLUTION

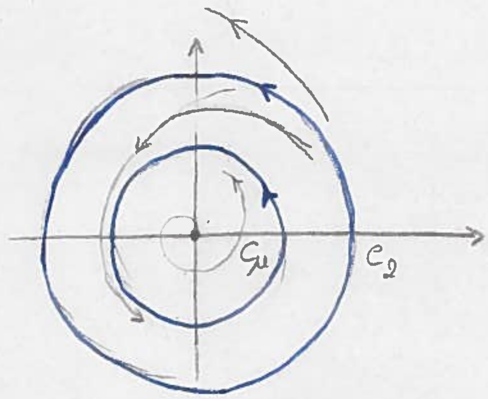
$\Rightarrow$ THE PHASE DIAGRAM WITH ORIENTATIONS IS SYMMETRIC WRT THE y-AXIS

3) SYSTEM 1: $\dot{r} = r(r-\mu)(r-2) = f(r)$, $\dot{\theta} = 1$

THE ORIGIN IS THE ONLY EP

$\mu < 2$: $f(r)$



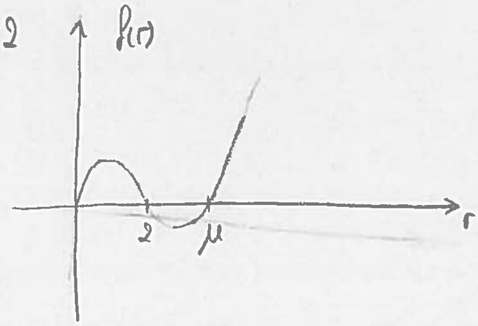


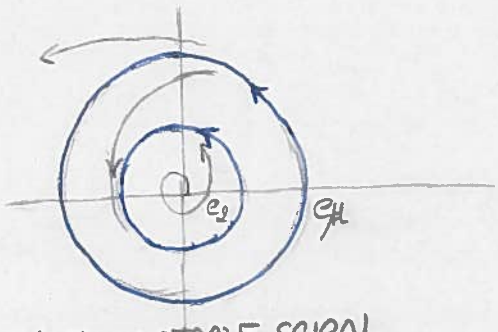
$\mathcal{C}_\mu$ STABLE LIMIT CYCLE

$\mathcal{C}_2$ UNSTABLE LIMIT CYCLE

$(0,0)$ UNSTABLE SPIRAL

$\mu > 2$ $f(r)$





$(0,0)$ UNSTABLE SPIRAL

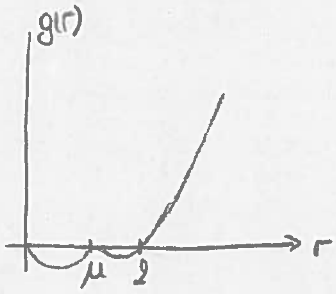
$\mathcal{C}_2$ .. STABLE LIMIT CYCLE

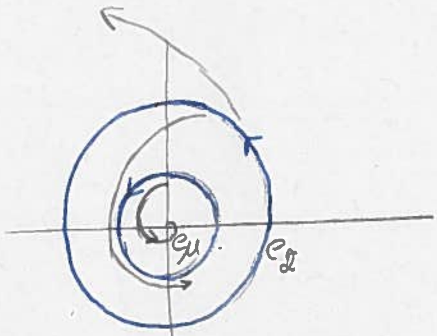
$\mathcal{C}_\mu$ .. UNSTABLE LIMIT CYCLE

$\Rightarrow \mu = 2$.. BIFURCATION POINT

SYSTEM 2: $\dot{r} = r(r-\mu)^2(r-2) = g(r), \; \dot{\theta} = 1$

$\mu < 2$:



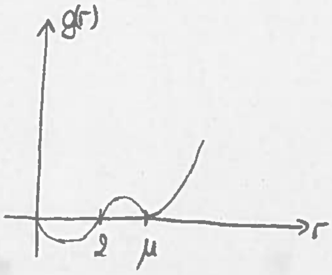


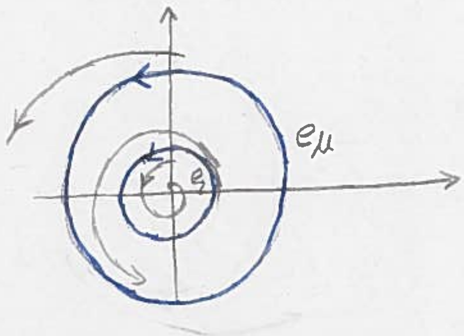
(0,0) STABLE SPIRAL

e_μ.. UNSTABLE LIMIT CYCLE

e_2.. UNSTABLE LIMIT CYCLE

$\mu > 2$





(0,0) STABLE SPIRAL

e_2.. UNSTABLE LIMIT CYCLE

e_μ .. UNSTABLE LIMIT CYCLE

$\Rightarrow$ NO BIFURCATION POINT

4) a) ·) EP

$$\left.\begin{aligned} \dot{x} = 0 &\Rightarrow y = 2 - x \\ \dot{y} = 0 &\Rightarrow y = x^2 + 2x - 2 \end{aligned}\right\} \Rightarrow x^2 + 2x - 2 = 2 - x \Rightarrow x^2 + 3x - 4 = 0$$

$$x_{1,2} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{16}{4}} = -\frac{3}{2} \pm \frac{5}{2} \lessgtr \begin{matrix} 1 \\ -4 \end{matrix}$$

EP $(-4,6)$, $(1,1)$

·) LIN: $J_{(x,y)} = \begin{pmatrix} 1 & 1 \\ 2x+2 & -1 \end{pmatrix} \Rightarrow$ ·) $J(1,1) = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$

EIGENVALUES $(1-\lambda)(-1-\lambda) - 4 = 0$

$\lambda^2 - 1 - 4 = 0$

$\lambda_{1,2} = \mp\sqrt{5}$

$\Rightarrow$ SADDLE

EIGENVECTORS $\begin{pmatrix} -1 \\ 1-\lambda_{1,2} \end{pmatrix}$

$$\cdot J_{(-4,6)} = \begin{pmatrix} 1 & 1 \\ -6 & -1 \end{pmatrix}$$

EIGENVALUES: $(1-\lambda)(-1-\lambda)+6=0$

$$\lambda^2 - 1 + 6 = 0$$
$$\lambda^2 = -5$$
$$\lambda_{1,2} = \pm i\sqrt{5}$$

CENTRE ?

$$\dot{x} = f(x,y) = x+y-2$$
$$\dot{y} = g(x,y) = x^2+2x-y-2$$

$\Rightarrow f_x(x,y) = 1$, $g_y(x,y) = -1$ $\Rightarrow f_x(x,y) = -g_y(x,y)$

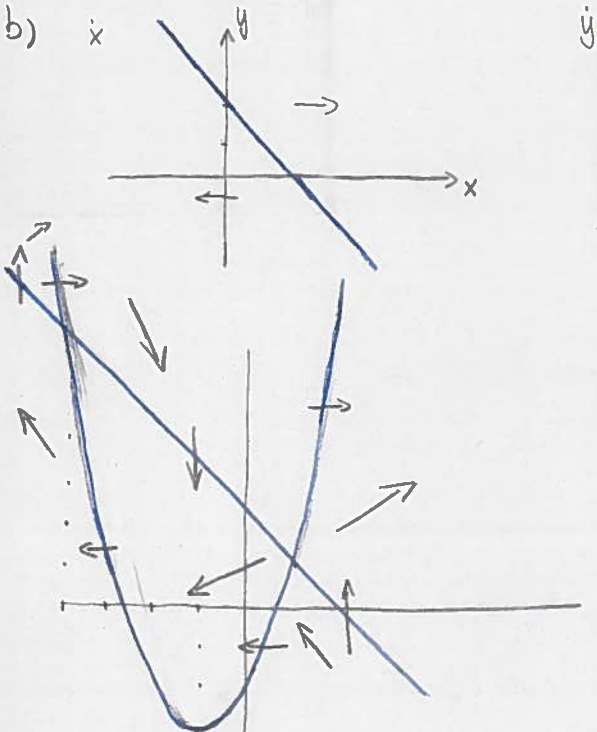
$\Rightarrow$ HAMILTONIAN SYSTEM.

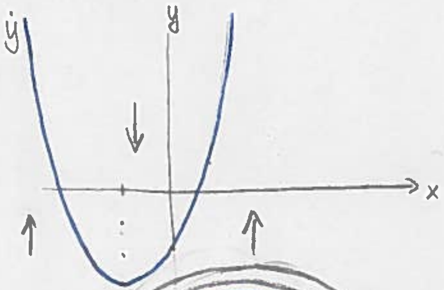
$\Rightarrow (-4,6)$ IS A CENTRE

$\Rightarrow (1,1)$.. SADDLE POINT $\Rightarrow$ UNSTABLE

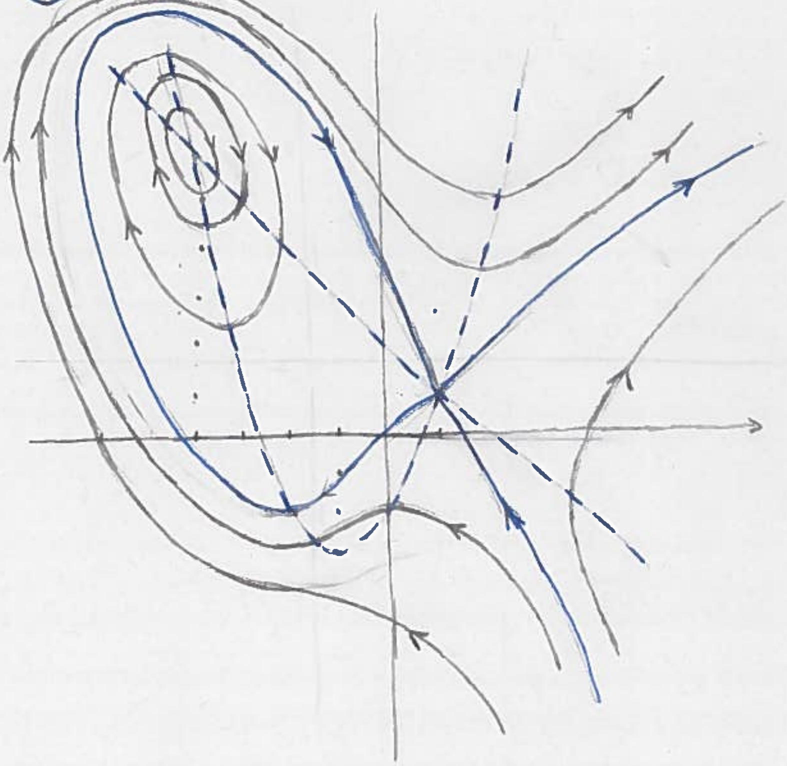
$(-4,6)$.. CENTRE $\Rightarrow$ (LIAPUNOV) STABLE, BUT NOT ASYMPTOTICALLY STABLE

b)





$\Gamma\ x^2+2x-2 = (x+1)^2 - 3 = y(x)$



5) EP: $\dot{x} = x(x^2+y^2)$
$\dot{y} = y(y^2+x^2)$ $\Rightarrow$ $(0,0)$ ONLY EP.

LET $V(x,y) = x^2+y^2$

$$\Rightarrow V(x,y)^\circ = 2x\dot{x}+2y\dot{y} = 2x^4+2y^4 \geq x^4+2x^2y^2+y^4 = (x^2+y^2)^2 = V(x,y)^2$$

$\Rightarrow$ V SATISFIES: $\frac{d}{dt}V(x(t),y(t)) \geq V(x(t),y(t))^2$

$$\Rightarrow \text{IF } V(x(0),y(0)) \neq 0: \; -\frac{1}{V(x(t),y(t))} + \frac{1}{V(x(0),y(0))} \geq t$$

$$\Rightarrow V(x(t),y(t)) - V(x(0),y(0)) \geq t\, V(x(t),y(t))V(x(0),y(0))$$

$$\Rightarrow V(x(t),y(t)) \geq \frac{V(x(0),y(0))}{1-tV(x(0),y(0))}$$

$$\downarrow$$

$$\infty \text{ IF } t \to \frac{1}{V(x(0),y(0))}$$

$\Rightarrow V(x(t),y(t)) \to \infty$ IN FINITE TIME IF $V(x(0),y(0)) \neq 0$

$\Rightarrow \|(x(t),y(t))\| \to \infty$ IN FINITE TIME IF $(x(0),y(0)) \neq (0,0)$

$\Rightarrow (x(t),y(t)) = (0,0)$ $\forall t$ IS THE ONLY GLOBAL SOLUTION

$\Rightarrow (x(0),y(0)) = (0,0)$ IS THE ONLY PAIR FOR WHICH THERE EXISTS A GLOBAL SOLUTION

6) THE SYSTEM IS OF THE FORM:

$$\begin{pmatrix} x \\ y \end{pmatrix}^\circ = \underbrace{\begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{e^{-t}}{1+t^2} & te^{-t^2} \\ \frac{1}{1+t^2} & 0 \end{pmatrix}}_{C(t)}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\cdot)$ $A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$ $\rightsquigarrow$ EIGENVALUES $\lambda_{1,2} = -1 \pm 3i$

$\Rightarrow$ ALL SOLUTIONS TO $\begin{pmatrix} x \\ y \end{pmatrix}^\circ = A\begin{pmatrix} x \\ y \end{pmatrix}$

ARE ASYMPTOTICALLY STABLE.

·) $C(t) = \begin{pmatrix} \frac{e^{-t}}{1+t^2} & te^{-t^2} \\ \frac{1}{1+t^2} & 0 \end{pmatrix} \Rightarrow \|C(t)\| \le \left|\frac{e^{-t}}{1+t^2}\right| + |te^{-t^2}| + \frac{1}{1+t^2}$

·) $\displaystyle\int_0^\infty \frac{1}{1+t^2}\,dt \le \int_0^1 1\,dt + \int_1^\infty \frac{1}{t^2}\,dt$

$\displaystyle = 1 - \frac{1}{t}\Big|_1^\infty = 2 < \infty$

·) $\displaystyle\int_0^\infty \frac{e^{-t}}{1+t^2}\,dt \le \int_0^\infty \frac{1}{1+t^2}\,dt < \infty$

·) $\displaystyle\int_0^\infty te^{-t^2}\,dt = -\frac{1}{2}\int_0^\infty (-2t)e^{-t^2}\,dt = -\frac{1}{2}e^{-t^2}\Big|_0^\infty = \frac{1}{2} < \infty$

$\displaystyle\Rightarrow \int_0^\infty \|C(t)\|\,dt < \infty$

$\Rightarrow$ THE ORIGIN IS ASYMPTOTICALLY STABLE & HENCE ALSO (LIAPUNOV) STABLE.

y) a) NOTE $\dot{x} = x\left(1 - \frac{1}{1+\varepsilon^2}x\right)$ $\Rightarrow$ TO EP $x=0$ & $x = 1+\varepsilon^2 > 1$

$\Rightarrow$ IF THE SOLUTION IS UNIQUE THEN

$0 \le x(t) \le 1+\varepsilon^2$

LET $f(x) = x - \frac{1}{1+\varepsilon^2}x^2 \Rightarrow f'(x) = 1 - \frac{2}{1+\varepsilon^2}x$

$\displaystyle\Rightarrow |f(x)-f(y)| \le \left|\int_y^x 1 - \frac{1}{1+\varepsilon^2}s\,ds\right| \le \max_{s\in\{x,y\}} \left|1 - \frac{1}{1+\varepsilon^2}s\right| |y-x|$

$\Rightarrow f$ ONLY LOCALLY LIPSCHITZ

$\Rightarrow f$ LIPSCHITZ ON $[-1-\varepsilon^2, 2+2\varepsilon^2]$ WITH LIPSCHITZ CONSTANT

$L = 2$, IE

$|f(x)-f(y)| \le 2|x-y|. \quad \forall x,y \in [-1-\varepsilon^2, 2+2\varepsilon^2]$

ASSUME THERE EXIST TWO SOLUTIONS $x(t) \neq y(t)$ TO $\dot{x}=f(x)$ WITH $x(0)=x_0 \in [0, 1+\varepsilon^2]$

LET $z(t)=(x(t)-y(t))^2$

$$\Rightarrow |\dot{z}(t)| = 2|(x(t)-y(t))(\dot{x}(t)-\dot{y}(t))|$$

$$= 2|x(t)-y(t)|\,|f(x(t))-f(y(t))|$$

$$\leq 4z(t)$$

AS LONG AS BOTH $x(t), y(t) \in [-1-\varepsilon^2, 2+2\varepsilon^2]$

$$\Rightarrow -4 \leq \frac{\dot{z}(t)}{z(t)} \leq 4 \quad \text{AS LONG AS BOTH } x(t), y(t) \in [-1-\varepsilon^2, 2+2\varepsilon^2]$$

$$\Rightarrow \quad z(t) \leq z(0)e^{4|t|} \quad \text{AS LONG AS BOTH } x(t), y(t) \in [-1-\varepsilon^2, 2+2\varepsilon^2].$$

$$\Rightarrow \quad z(t)=0 \text{ IF } z(0)=0 \text{ AS LONG AS BOTH } x(t), y(t) \in [-1-\varepsilon^2, 2+2\varepsilon^2]$$

$$\Rightarrow \quad x(t)=y(t) \text{ AS LONG AS BOTH } x(t), y(t) \in [-1-\varepsilon^2, 2+2\varepsilon^2] \quad ↯$$

- IF $x(0)=0 \Rightarrow$ UNIQUE SOL $x(t) \equiv 0$
- IF $x(0)=1+\varepsilon^2 \Rightarrow$ UNIQUE SOL $x(t) \equiv 1+\varepsilon^2$

$\Rightarrow \exists$ UNIQUE SOL $x(t)$ TO $\dot{x}=f(x)$ WITH $x(0)=\frac{1}{2}$

AND $0 \leq x(t) \leq 1+\varepsilon^2 \ \forall t$.

b) FROM a) $0 \leq x^0(t) \leq 1$

$0 \leq x^\varepsilon(t) \leq 1+\varepsilon^2$

LET $G(t)=(x^0(t)-x^\varepsilon(t))^2$ AND $f_\varepsilon(x)=x-\frac{1}{1+\varepsilon^2}x^2$

THEN $|\dot{G}(t)| = 2|x^0(t)-x^\varepsilon(t)|\,|\dot{x}^0(t)-\dot{x}^\varepsilon(t)|$

$$= 2|x^0(t)-x^\varepsilon(t)|\,|f_0(x^0(t))-f_\varepsilon(x^\varepsilon(t))|$$

$$\leq 2|x^0(t)-x^\varepsilon(t)|\left(|f_0(x^0(t))-f_\varepsilon(x^0(t))|+|f_\varepsilon(x^0(t))-f_\varepsilon(x^\varepsilon(t))|\right)$$

$$\leq 2|x^0(t)-x^\varepsilon(t)|\left|\frac{1}{1+\varepsilon^2}-1\right|x^0(t)^2 + 2|x^0(t)-x^\varepsilon(t)|\max_{s\in[0,1+\varepsilon^2]}\left|1-\frac{1}{1+\varepsilon^2}2s\right||x^0(t)-x^\varepsilon(t)| \leftarrow \text{SEE a)}$$

$$\leq 2|x^0(t)-x^\varepsilon(t)|\varepsilon^2 + 2|x^0(t)-x^\varepsilon(t)|^2$$

$$\leq \varepsilon^4 + 3|x^0(t)-x^\varepsilon(t)|^2$$

$$= \varepsilon^4 + 3G(t)$$

$$\Rightarrow -1 \leq \frac{\dot{G}(t)}{\varepsilon^4+3G(t)} \leq 1 \quad \forall t$$

$$\Rightarrow \quad 3G(t) \leq \varepsilon^4+3G(t) \leq (\varepsilon^4+3G(0))e^{3|t|}$$

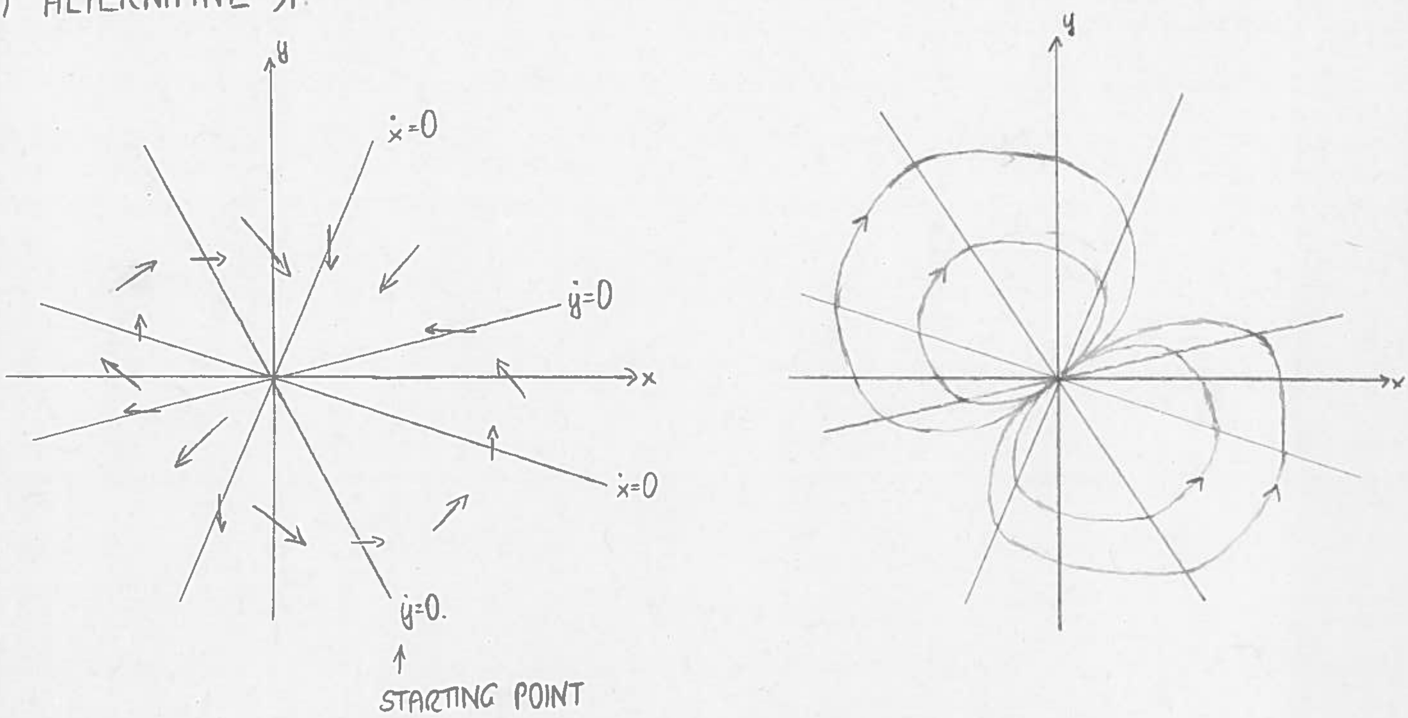
$$3G(t) \leq \varepsilon^4 e^{3|t|}$$

$$G(t) \leq \frac{1}{3}\varepsilon^4 e^{3|t|}$$

HERE WE USED $G(0)=(x^0(0)-x^\varepsilon(0))^2=\left(\frac{1}{2}-\frac{1}{2}\right)^2=0$

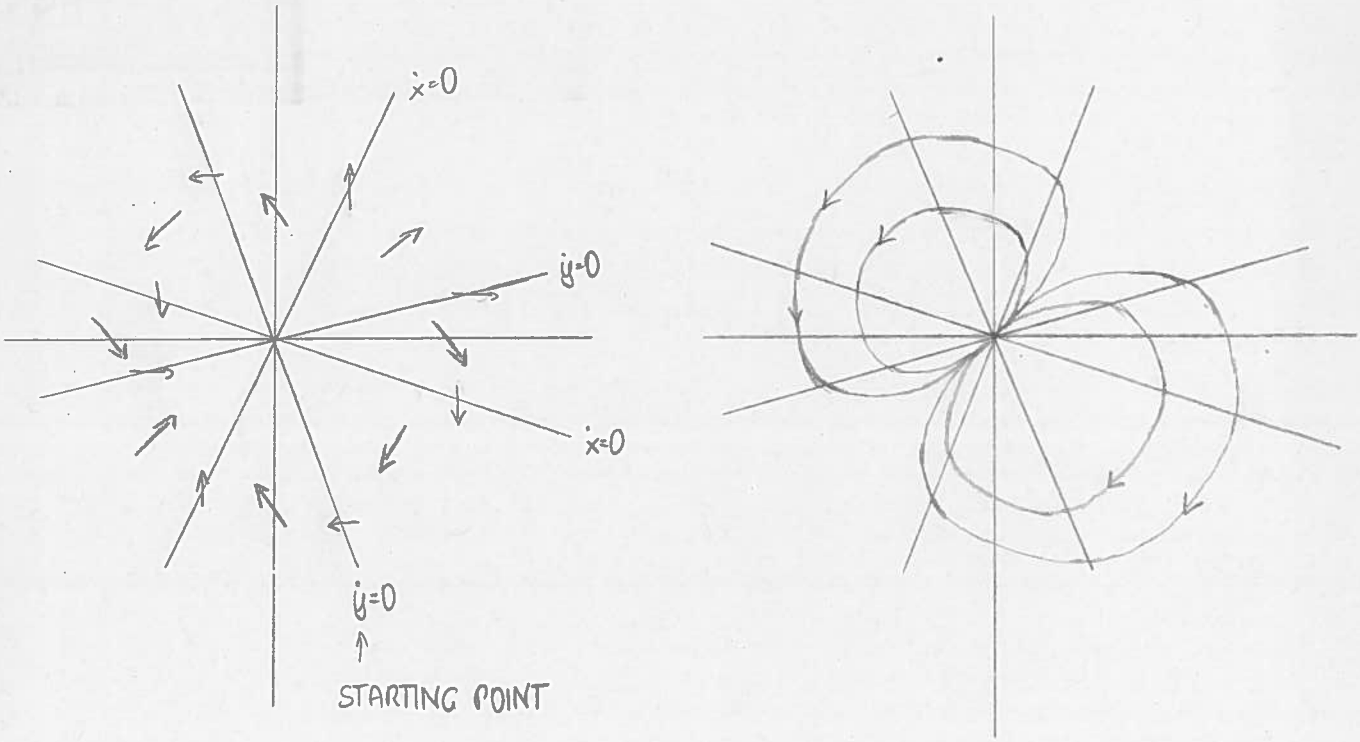
$$\Rightarrow |x^0(t) - x^\varepsilon(t)| \le \frac{1}{\sqrt{3}} e^{\frac{3}{2}|t|} \varepsilon^2 = K(t)\varepsilon^2$$

8) ALTERNATIVE 1



$\to \rightsquigarrow \uparrow \rightsquigarrow \leftarrow \rightsquigarrow \downarrow \rightsquigarrow \to \rightsquigarrow \uparrow \rightsquigarrow \leftarrow \rightsquigarrow \downarrow \rightsquigarrow \to \Rightarrow$ INDEX 2

ALTERNATIVE 2



$\leftarrow \rightsquigarrow \downarrow \rightsquigarrow \to \rightsquigarrow \uparrow \rightsquigarrow \leftarrow \rightsquigarrow \downarrow \rightsquigarrow \to \rightsquigarrow \uparrow \rightsquigarrow \leftarrow \Rightarrow$ INDEX 2