Problem 1 Sketch the phase diagram, with orientations, of the system

$$\dot{x} = x + 9y,$$
$$\dot{y} = 6x + 4y.$$

Problem 2 For each of the following systems, determine whether the origin is a stable, asymptotically stable, or unstable equilibrium. If it is stable or asymptotically stable, find a weak or strong Lyapunov function.

a. $$\dot{x} = -2x^2y + 2y^3,$$
$$\dot{y} = -x^3 + 2xy^2.$$

b. $$\dot{x} = -2x^3 + xy^2,$$
$$\dot{y} = x^3 - y^3.$$

c. $$\dot{x} = -y - xz^2,$$
$$\dot{y} = x - yz^2,$$
$$\dot{z} = -x^2y^2z.$$

Problem 3 Find all bifurcation points of the system

$$\dot{x} = \mu x - y^2,$$
$$\dot{y} = x + y - 1.$$

Sketch a bifurcation diagram using coordinates (μ, x), indicating the type of each equilibrium point in the diagram.

Due to a miscalculation while putting this problem set together, a small part of this problem (telling nodes and spirals apart) requires solving a fourth degree equation with no easy solutions. Future students doing this problem are adviced to either skip that part, or to resort to numerical calculations at that point. (This is normal in real-world examples; not so much in exam problems.)

Problem 4 Consider two plane vector fields

$$\big(X(x,y), Y(x,y)\big) \text{ and } \big(\widetilde{X}(x,y), \widetilde{Y}(x,y)\big)$$

satisfying

$$(X - \widetilde{X})^2 + (Y - \widetilde{Y})^2 < X^2 + Y^2 \quad \text{on a simple, closed curve } \Gamma.$$

Explain briefly why Γ has the same index with respect to the two vector fields.

Hint: Draw a picture. How large can the angle between the two vectors get?

Problem 5 Consider the dynamical system

$$\dot{x} = x^2 - y^4,$$
$$\dot{y} = y^2 - x^4.$$

a. What are the equilibrium points of the system? Without computing the index of each equilibrium point, compute the sum of the indices of all of them.

b. Show that the reflection through either of the diagonal lines given by $x = y$ or $x + y = 0$, respectively, maps each phase path of the system to another phase path. For each of the two reflections, state whether it preserves or reverses the direction of the phase paths.

c. Classify each of the equilibrium points, and compute the index for each one. Two of them are centres. You only need to show that their linearisations are centres. (The fact that they are centres for the full system follows from the symmetries in **b**, but you don't need to prove that.) Is the equilibrium point at the origin unstable, stable, or asymptotically stable?

d. Sketch the phase diagram with orientations.

You may take the following as known without proof: Any unbounded forward phase path must enter the region $\{(x, y) \mid -y^2 < x < 0 \text{ and } -x^2 < y < 0\}$.