TMA4165 Differential equations and dynamical systems 2019-06-01

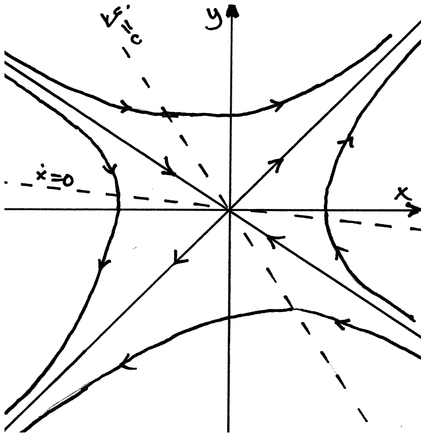
Solution

Problem 1

This is a linear system, whose matrix has trace $p = 1 + 4 = 5$ and determinant $q = 1{\cdot}4 - 6{\cdot}9 = -50 < 0$, so we have a saddle. The characteristic polynomial of the matrix is $\lambda^2 - 5\lambda - 50 = (\lambda + 5)(\lambda - 10)$, and the eigenvalues are -5 and 10.

An eigenvector for $\lambda = -5$ is $(3, -2)$, and an eigenvector for $\lambda = 10$ is $(1, 1)$.

The dashed lines in the figure are the *nullclines*, defined by $\dot{x} = 0$ and $\dot{y} = 0$. They serve as a guide to slightly better accuracy. (Since the two eigenvalues have different absolute values, the phase paths are not hyperbolas, lacking their symmetry.)

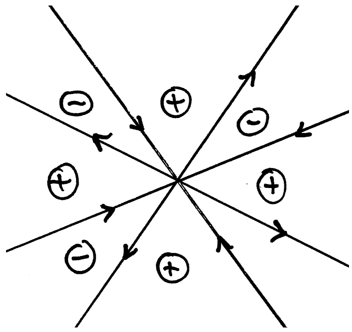


Problem 2

a. The system is Hamiltonian with $H(x, y) = \frac{1}{4}x^4 - x^2y^2 + \frac{1}{2}y^4$. We can factor this as follows:

$$\begin{aligned} H(x,y) &= \tfrac{1}{4}(x^2 - 2y^2)^2 - \tfrac{1}{2}y^4 \\ &= \tfrac{1}{4}\big(x^2 - (2+\sqrt{2})y^2\big)\big(x^2 - (2-\sqrt{2})y^2\big). \end{aligned}$$

The figure shows a sign diagram resulting from this, with the flow along the lines $x = \pm(2 \pm 2^{1/2})^{1/2}y$ as indicated by the arrows. The existence of invariant rays with an outward flow shows that the origin is unstable.



Alternatively we can look for solutions satisfying $x = ay$ for some constant a. This is natural, since the right hand side of the system is homogeneous (of degree 3). This leads to $\dot{x} = a\dot{y}$, $\dot{x} = (-2a^2 + 2)y^3$, and $\dot{y} = (-a^3 + 2a)y^3$, which in turns yields the equation $a^4 - 4a^2 + 2 = 0$, with solutions $a^2 = 2 \pm 2^{1/2}$. Two of the possible values for a give $-a^3 + 2a > 0$, making the equation $\dot{y} = (-a^3 + 2a)y^3$ unstable.

b. Asymptotically stable with strong Lyapunov function $V(x, y) = x^2 + y^2$. In detail:

$$\begin{aligned} \tfrac{1}{2}\dot{V} &= x\dot{x} + y\dot{y} = -2x^4 + x^2y^2 + x^3y - y^4 \\ &\le -2x^4 + x^2y^2 + \tfrac{1}{2}x^2(x^2+y^2) - y^4 && xy \le \tfrac{1}{2}(x^2+y^2) \\ &= -\tfrac{3}{2}x^4 + \tfrac{3}{2}x^2y^2 - y^4 \\ &\le -\tfrac{3}{2}x^4 + \tfrac{3}{4}(x^4+y^4) - y^4 && x^2y^2 \le \tfrac{1}{2}(x^4+y^4) \\ &= -\tfrac{3}{4}x^4 - \tfrac{1}{4}y^4 < 0 \quad \text{for } (x,y) \ne (0,0). \end{aligned}$$

c. Stable with weak Lyapunov function $V(x, y, z) = x^2 + y^2 + z^2$. In detail:

$$\tfrac{1}{2}\dot{V} = x\dot{x} + y\dot{y} + z\dot{z} = -xy - x^2z^2 + xy - y^2z^2 - x^2y^2z^2 \le 0 \quad \text{for } (x,y,z) \ne (0,0,0).$$

The system is not asymptotically stable, for the plane $z = 0$ is invariant, and the flow there is simple rotation around the origin.

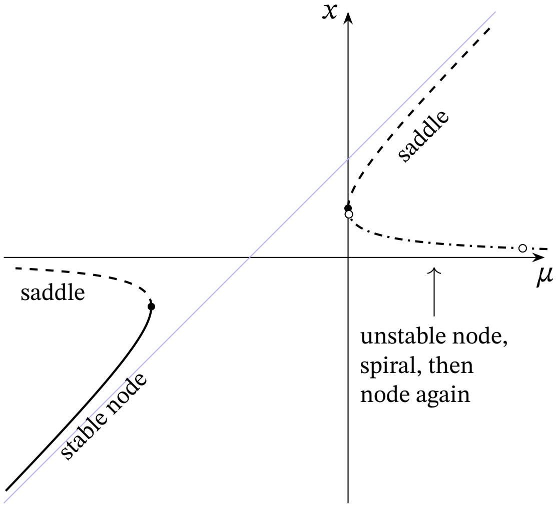
Problem 3

To find the equilibrium points, insert $x = 1 - y$ from the equation $\dot{y} = 0$ into the equation $\dot{x} = 0$, resulting in $y^2 + \mu y - \mu = 0$. Thus we get the two solutions $(x_\pm, y_\pm)$ where

$$\left.\begin{aligned} y_\pm &= \tfrac{1}{2}\big(-\mu \pm \sqrt{\mu^2 + 4\mu}\big) \quad \text{and} \\ x_\pm &= 1 + \tfrac{1}{2}\big(\mu \mp \sqrt{\mu^2 + 4\mu}\big) \end{aligned}\right\} \qquad \text{when } \mu \geq 0 \text{ or } \mu \leq -4.$$

(The two solutions coincide when $\mu = 0$ or $\mu = -4$. There are no solutions when $-4 < \mu < 0$.)

The bifurcation diagram is shown on the right. The calculations leading up to it are as follows.



The saddle–node bifurcations are marked with filled circles; the transitions between node and spiral are marked with open circles.

The matrix of the linearisation at each equilibrium point is

$$\begin{pmatrix} \mu & -2y_\pm \\ 1 & 1 \end{pmatrix}.$$

Its trace is $p = \mu + 1$, and its determinant is $q_\pm = \mu + 2y_\pm = \pm\sqrt{\mu^2 + 4\mu}$. Thus (x_-, y_-) is a saddle point.

To analyse (x_+, y_+), note that it will be a node if $p^2 > 4q_+$. This is equivalent to $(\mu+1)^4 > 16(\mu^2 + 4\mu)$. That is true at $\mu = 0$ and $\mu = -4$, where the right hand side vanishes. Therefore, it is also true for μ sufficiently close to those values. It turns out to hold for all $\mu < -4$, and for $\mu > 0$ except for $\mu \in [0.017, 3.55]$ (approximately). In the interior of the named interval, we get a spiral instead. The node or spiral is unstable when $\mu > 0$ and unstable when $\mu < -4$.

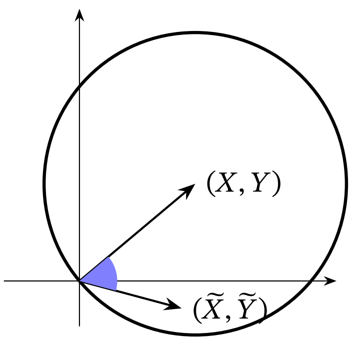
Due to a miscalculation while putting the exam together, this detail turned out much too hard for hand calculation.[1] Needless to say, this will be taken into account while grading.

To sum it up, the system has two saddle–node bifurcations, one at $\mu = 0$ and one at $\mu = -4$.

For the bifurcation diagram, substitute $y = 1 - x$ (from $\dot{y} = 0$) into $\mu x = y^2$ (from $\dot{x} = 0$) to get $\mu x = (1-x)^2$. Rewrite as $x^2 - (\mu+2)x + 1 = 0$ and complete the square to get $(x - \frac{\mu}{2} - 1)^2 - (\frac{\mu}{2} - 1)^2 + 1 = 0$ This is the equation of a hyperbola with asymptotes $x - \frac{\mu}{2} - 1 = \pm(\frac{\mu}{2} + 1)$, i.e., $x = 0$ and $x = \mu - 2$ (the light blue line in the diagram). You may prefer instead to write the equation in the form $\mu + 2 = x + 1/x$.

Problem 4

As the figure indicates, the angle between the two vector fields is less than $\pi/2$, because $(\widetilde{X}, \widetilde{Y})$ must lie within the circle shown.



Thus, if φ is the continuously varying angle between the x axis and (X, Y), and similarly for $\tilde{\varphi}$ and $(\widetilde{X}, \widetilde{Y})$, then $|\varphi - \tilde{\varphi}| < \pi/2$ everywhere on Γ. (Strictly speaking, we can *choose* the angles so that this holds initially, and then it will remain true along Γ by continuity.) Writing $\Delta\varphi$ for the total change in φ along Γ, similarly for $\Delta\tilde{\varphi}$, it follows that $|\Delta\varphi - \Delta\tilde{\varphi}| < \pi$. Since both $\Delta\varphi$ and $\Delta\tilde{\varphi}$ are integer multiples of 2π, the difference must be zero, so the two indices are the same.

[1]Thanks to Øyvind Steensland, who pointed out the mistake in an earlier version of the solutions.

Problem 5

a. The equilibrium points are $(0,0)$, $(1,1)$, $(-1,-1)$, $(1,-1)$ and $(-1,1)$.

On a large circle centred at the origin, $(\dot{x},\dot{y})$ can never enter the first quadrant: For if $\dot{x} > 0$ and $\dot{y} > 0$ then $x^2 > y^4 > x^8$, implying that $|x| < 1$, and similarly $|y| < 1$. Therefore the index of the large circle is zero, and therefore so is the sum of indices of the equilibrium points within.

b. The reflection through the line $x = y$ maps (x,y) to (y,x). If $\big(x(t),y(t)\big)$ is a solution, then so is $\big(y(t),x(t)\big)$. Thus one phase path is mapped to the other, with orientations preserved.

The reflection through the line $x + y = 0$ maps (x,y) to $(-y,-x)$. If $\big(x(t),y(t)\big)$ is a solution, then so is $\big(-y(-t),-x(-t)\big)$. Thus one phase path is mapped to the other, with orientations reversed.

c. The linearisation of the system at each equilibrium point (x,y) is given by the matrix

$$A = \begin{pmatrix} 2x & -4y^3 \\ -4x^3 & 2y \end{pmatrix}.$$

At the equilibrium $(0,0)$, all entries of A are zero; hence this equilibrium defies simple classification.

At the equilibrium $(1,1)$,

$$A = \begin{pmatrix} 2 & -4 \\ -4 & 2 \end{pmatrix}.$$

Its trace is $p = 4$, and its determinant is $q = -12 < 0$, hence $(1,1)$ is a saddle point. The eigenvectors are $(1,1)$, with eigenvalue -2, and $(1,-1)$, with eigenvalue 6.

The equilibrium at $(-1,-1)$ is also a saddle point, and the eigenvectors are $(1,1)$, with eigenvalue 2, and $(1,-1)$, with eigenvalue -6. (This follows by the symmetry around the line $x + y = 0$, or a similar computation as the one above.)

At the equilibrium $(1,-1)$,

$$A = \begin{pmatrix} 2 & 4 \\ -4 & -2 \end{pmatrix}.$$

Its trace is $p = 0$, and its determinant is $q = 12$. Hence the linearisation is a centre. As stated in the problem, this together with the symmetry around the line $x + y = 0$ implies that the point is a centre for the full system. The rotation direction is clockwise.

The equilibrium $(-1,1)$ is also a centre, by the symmetry around the line $x = y$. The rotation direction is counterclockwise.

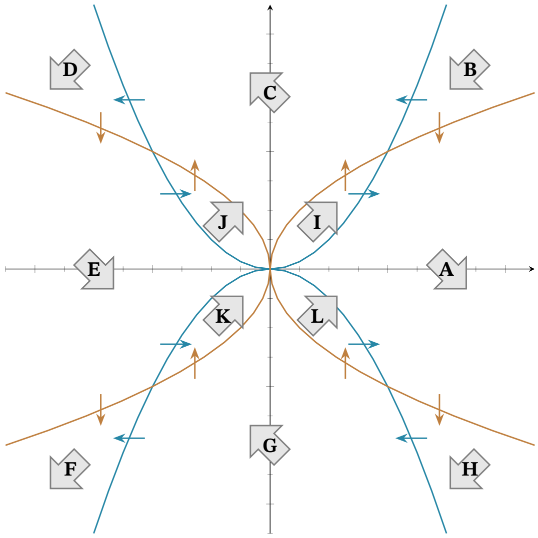
The equilibria at $(1,1)$ and $(-1,-1)$, being saddle points, have index -1.

The equilibria at $(1,-1)$ and $(-1,1)$, being centres, have index $+1$.

Since the sum of all indices is 0, and the indices computed above have sum 0, the remaining equilibrium $(0,0)$ has index 0 as well.

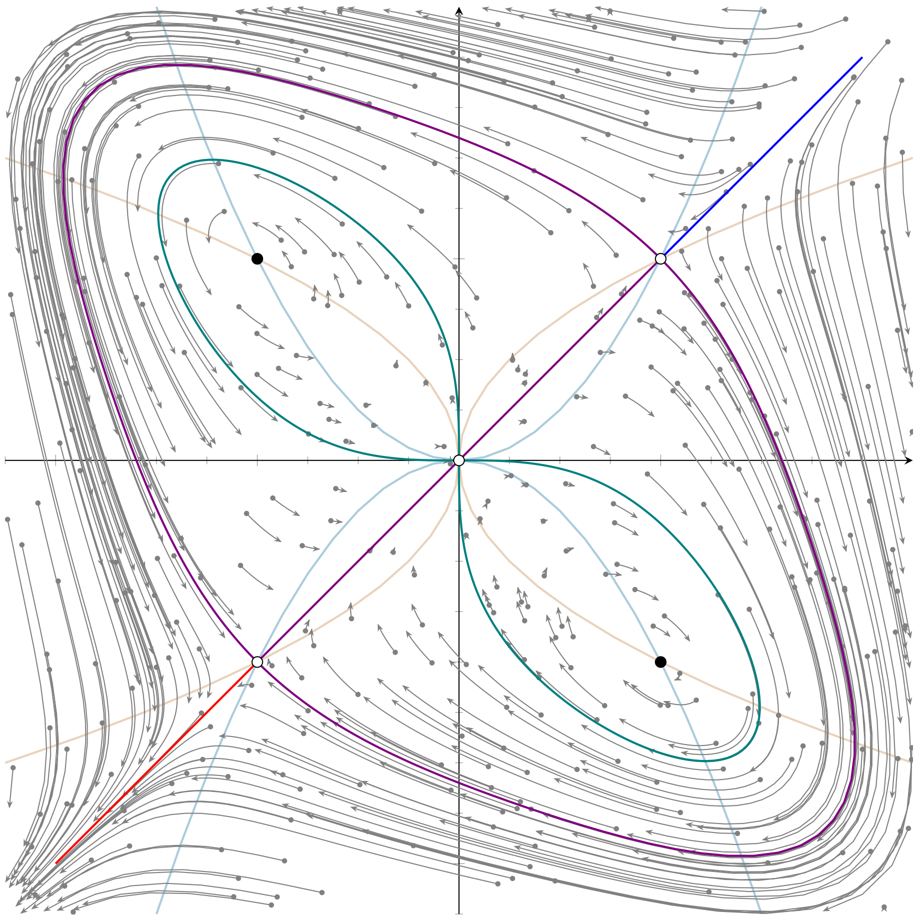
The equilibrium at the origin is unstable. This is most easily seen by noting that if x solves $\dot{x} = x^2 - x^4$, then $\big(x(t),x(t)\big)$ is a solution of the full system. In particular, any solution starting at (x,x) with $x > 0$ and $x \approx 0$ will follow the diagonal to $(1,1)$.

d. To get started, here is a picture showing the x- and y-nullclines, defined respectively by $\dot{x} = 0 \;(x = \pm y^2)$ and $\dot{y} = 0 \;(y = \pm x^2)$. The nullclines are equipped with arrows showing the direction of the flow at each segment. The nullclines divide the plane into twelve regions, named A–L, and the general direction of the flow is indicated by a fat arrow in each region.



The region named in the question, $\{(x,y) \,|\, -y^2 < x < 0 \text{ and } -x^2 < y < 0\}$, is the one labeled **F**. The stated property on unbounded forward phase paths is important in drawing the phase diagram, because it implies that no trajectory can escape to infinity within either of the regions A or C. By the symmetry about $x + y = 0$, no trajectory escapes to infinity going *backward* in time from regions E or G. The other unbounded regions are easier to deal with: In particular, any trajectory in region D must continue into region E, and from there further into F, K, or J (except for some separatrices converging to $(-1,-1)$ or the origin).

This picture is computer generated. It is of course unrealistic (not to mention *unfair*) to expect every aspect to be present in a hand drawn version, but the more the merrier!



The grey curve pieces are partial phase paths. They have a circle at the start, and an arrow at the end, thus indicating direction. Their lengths are proportional to average flow speed. The two stable equilibrium points (centres) are filled black circles, and the three unstable ones are white circles. Special orbits – separatrices – are colour coded: They are four heteroclinic and two homoclinic orbits, plut stable and unstable manifolds of the saddles at $(1,1)$ and $(-1,-1)$ respectively. The heteroclinic and homoclinic orbits exists largely due to the symmetries of the system.

The nullclines are also included for reference.