Home assignment to week 3

1. Prove the triangle inequality:
$$\|f+g\| \le \|f\| + \|g\|.$$
You can do it either for functions in $L^2(0,1)$ or for general Hilbert spaces.

2. Consider the so-called Haar system of functions in $L^2(0,1)$: $e_0^0(x) = 1$,
$$e_n^k(x) = \begin{cases} 2^{n/2}, & \text{for } \frac{k-1}{2^n} \le x < \frac{k-1/2}{2^n} \\ -2^{n/2}, & \text{for } \frac{k-1/2}{2^n} \le x < \frac{k}{2^n} \\ 0 & \text{othwerwise.} \end{cases}$$
$n = 1, 2, \dots,\ k = 1, 2, \dots, n$. Prove that this system is an orthonormal basis in $L^2(0,1)$.

3. Given a sequence of complex numbers $\{z_n\}_1^\infty$ such that the limit
$$w = \lim_{n\to\infty} z_n \qquad (*)$$
exists, prove that
$$\frac{1}{n}\sum_1^n z_k \to w. \qquad (**)$$
Give an example of a sequence $\{z_n\}_1^\infty$ such that $(*)$ does not hold, yet $(**)$ is still true.

4. Given a function $f \in L^2(-\pi,\pi)$ prove that
$$\left\| \frac{1}{2n-1} \sum_{-n+1}^{n-1} f(x + k\pi/n) - \hat{f}(0) \right\|_{L^2(-\pi,\pi)} /to0$$
as $n \to \infty$. *Hint:* Compute the Fourier coefficients of the sum and use the Parseval identity.

5. Find the Fourier series for the function $f(x) = \cosh ax$ in $L^2(-\pi,\pi)$