Lectures 2 - 3

Pointwise convergence, convolution

1. Rieman - Lebesgue lemma.

- words that terms of F. series should decay.
- Space $L^1(-\pi, \pi)$.
- Remind the Schwartz inequality.
- $L^2(-\pi, \pi) \subset L^1(-\pi, \pi)$
- Proof of the R.L lemma.
- Say that it is the same for trigonometrical series.
- Speculation about data compression.
- Spectra, frequencies.

2. Decay of coefficients for smooth functions

- periodic Functions with one derivative in L^2
- Schwartz inequality for sequences.
- (Uniform) Convergence of the Fourier series for periodic function with derivative in $L^2(-\pi, \pi)$
- Infinitely differentiable functions $\Leftrightarrow$ rapidly decaying coefficients.

3. Local convergence

- Representation with Dirichle kernel,
  expression for the Dirichlet kernel.
- Graph of the Dirichlet kernel. Waving hands explanation why the Fourier series is convergent if the function is nice.
- **Definition** Lip_α in an interval.
- **Without proof**: $Lip_\alpha \Rightarrow$ convergence of the Fourier series on an interval
- Theorem: If f_1 and f_2 coincide in an interval, then their Fourier series converges / diverges simultaneously.

4. Cezaro summation

- Description of the procedure
- Formula with Fejer kernel
- Expression for the Fejer kernel. $F_N(x) = \frac{1}{N+1}\left(\frac{\sin\frac{(N+1)x}{2}}{\sin x/2}\right)^2$
- Properties of the Fejer kernel:
  - graph.
  - $F_n \geq 0$
  - $\int_{-\pi}^{\pi} F_n = 1$
  - $\int_{|x|>\delta} F_n \to 0$, $n \to \infty$

- f - ~~perio~~ 2π-periodic and continuous $\Rightarrow$
  $\Rightarrow$ Cesaro summation reconstructs f.

5. Convolution

- Definition of convolution of periodic functions.
- What happens to Fourier coefficients.
- Convolution is smooth in case one of the functions is smooth.
- Approximative unit (you may think about the Fejer kernel)
- Convolution gives approximation in L^2 - norm:
  - Remind: Shifts are continuous in L^2 norm.
  - Proof of the approximation theorem

Emphasis the general pattern of proof

- Remark The same works for L^p, $1 \le p < \infty$.
- Statement The exponential system $\{e^{int}\}_{-\infty}^{\infty}$ spans the whole $L^2(-\pi, \pi)$.
- Waving hands: . Convolution is a numerical tool to obtain approximation by smooth function
  - It is more convenient to do it by multiplying the coefficients.
  - In the next lecture we discuss numerical way to calculate the coefficients.