TMA 4180 Optimeringsteori

Spring 2005/2010

Review Questions

This review list contains simple and somewhat more difficult questions from the curriculum. The first part of the list is based on a list from 2005, where the curriculum was slightly different f\rom this year.

You should cross out all questions you know the answer to already, and work on the rest until everything has been crossed out.

For some of the questions, the idea is just to look up the answer in the notes and trying to understand what is going on.

NB! There will be no solutions to the questions given here.

1. Explain the following terms for a real-valued function on a domain $\Omega \in \mathbb{R}^n$: A *local minimum*, a *global minimum*, and a *strict local minimum*.

2. How is a *closed* and a *bounded* set defined?

3. The continuous function f is defined on a closed and bounded set $\Omega \in \mathbb{R}^n$. Does the problem $\min_{x\in\Omega} f(x)$ have a solution? What about $\max_{x\in\Omega} f(x)$?

4. Which of the following problems have solutions:

$$
\begin{aligned}
y &= \arg \min_{-1<x<1} x^3, \\
y &= \arg \min_{0<x<1} (2x-1)^4, \\
y &= \arg \min_{x\in\mathbb{R}^n} \left\{x'Ax \ + b'x + \gamma\right\}, \ A > 0, \\
y &= \arg\min_{x\in\mathbb{R}^n, \|x\|\le 1} \ x'Ax \ , \ A \text{ arbitrary.}
\end{aligned}
$$

5. What is the difference between the *Taylor Formula* and the *Taylor Series*?

6. Let $g : \mathbb{R} \to \mathbb{R}$. Recall the 1-D *Taylor formula*

$$g(x) = g(0) + g'(0)\,x + \frac{1}{2} g''(x_\theta)\,x^2.$$

   What is the meaning of x_θ?

7. Let $f : \mathbb{R}^n \to \mathbb{R}$. How is the *gradient* and the *Hessian* of f defined, and how can we write the Taylor series of f to second order?

8. Use the 1-D Taylor formula above to derive

$$f(x) = f(x_0) + \nabla f(x_0)'(x - x_0) + \frac{1}{2}(x - x_0)'\nabla^2 f(x_\theta)(x - x_0).$$

9. How is the *directional derivative*, $\delta f(x; p)$ defined?

10. Prove that

$$\delta f(x; p) = \nabla f(x)\, p$$

    if the gradient of f exists and is continuous around x.

11. What are *feasible directions* from a point $x \in \Omega$?
12. Express the necessary *first order* condition for a feasible point x^* in order to be a local minimum. What is the condition when x^* is an *inner point* in Ω?
13. State the *second order* necessary conditions which needs to hold in an inner local minimum.
14. Which second order condition guarantee that we have a *strict* local minimum?
15. What is a *convex set*?
16. If $A_1, \cdots, A_n$ are convex sets, are $\cup_{i=1}^n A_i$ and $\cap_{i=1}^n A_i$ convex? Give examples and proofs.
17. What is a *convex function*? What is a *strictly convex* function?
18. Which of these functions are convex:

$$\begin{aligned}
&y = 2x - 5, \\
&y = |x| - 3x, \\
&y = x^{100} - \exp(x/100), \\
&y = \|x\|, \ x \in \mathbb{R}^n, \\
&y = x'Ax - 100c'x - 3, \ x, c \in \mathbb{R}^n, \ A \geq 0.
\end{aligned}$$

19. Prove that $\alpha f + \beta g$ is convex when f, g are convex and α and β are *what*?
20. Prove that if $f_1, \cdots, f_n$ are convex for all x, then $g(x) = \max(f_1(x), \cdots, f_n(x))$ is convex.
21. Prove that if $f_1, \cdots, f_n$ are convex, then

$$\mathcal{C} = \{x; f_1(x) \leq c_1, \cdots, f_n(x) \leq c_n\}$$

    is convex.
22. Prove that the sets

$$\begin{aligned}
&\{x\ ;\ Ax = b\}, \\
&\{x\ ;\ Ax \geq b\},
\end{aligned}$$

    are convex.
23. What is a *differentiable function.*
24. Define the *tangent plane* of a differentiable function.
25. Argue that the graph of a differentiable convex function is above its tangent planes.
26. What is the *big result* for minima of convex functions defined on convex sets?
27. What is a *line search* algorithm?
28. What is a *trust region* method?
29. Derive the gradient, the Hessian and the solution for the quadratic model problem $\min_{x \in \mathbb{R}^n} f(x)$, where

$$f(x) = a + b'x + \frac{1}{2}x'Ax, \ A > 0.$$

30. Derive *Newton's method.*

31. What is a *descent direction* and what are the angles between descent directions and the gradient?
32. What is the angle between the gradient and the search direction at the exact minimum of a line search step?
33. Explain what the *Strong Wolfe's Conditions* are for a one dimensional line search. Why are these conditions important?
34. What is N&W's definition of a *globally convergent algorithm*?
35. What can happen with globally convergent algorithms on infinite domains?
36. What is the content of *Zoutendijk's Theorem*?
37. State the *A-scalar product* and show that it defines a *norm* on $\mathbb{R}^n$.
38. Derive the *Steepest Descent* algorithm for the quadratic model problem.
39. What determines the *convergence rate* for the Steepest Descent algorithm?
40. Explain what is meant by *quadratic convergence* for Newton's method.
41. Which cases will occur in the solution of the Trust Region model problem

$$\min_{\|p\| \leq \Delta} \left(a'p + \frac{1}{2} p' B p \right),$$

    and how can it be solved?
42. How is Δ adjusted in the Trust Region method?
43. What is the *Cauchy point* and the *Dog-Leg method* in the Trust Region algorithm?
44. How is a *Hilbert space* defined?
45. Let H_0 is a closed subspace of the Hilbert space H and

$$y = \arg \min_{z \in H_0} \|x - z\|.$$

    What is y called and which properties does it have?
46. Let $\{e_j\}_{i=1}^n$ be a basis of H and $H_0 = \operatorname{span}\{e_1, \cdots, e_k\}$, $k < n$. How is y in the previous question related to the Fourier series of x?
47. Explain the two main ideas of the *Conjugate Gradient* method applied to the quadratic model problem.
48. Explain the idea of the *Preconditioned Conjugate Gradient*.
49. What is the *Polak-Rebiere* method?
50. Explain the main idea behind *Quasi Newton* methods?
51. What is a *triplet* for 1-d minimization problems?
52. What are the *Golden Ratio* and *Parabolic Fit* search for 1-d minima?
53. What is an *n-dimensional simplex* in the *Amoeba method*?
54. Explain the 4 basic operations in the Amoeba method.

55. (research required!) Explain the additional *outer contraction* used in the Matlab function `fminsearch`.

56. State the linear problem in *Least Square Optimization* and derive the *Normal Equations.*

57. Define the (reduced form) *Singular Value Decomposition* for a general matrix A.

58. Discuss the solutions of the normal equations in the *full rank* and the *rank deficient* cases.

59. What is the *Moore-Penrose* inverse?

60. What is the simplest regularization method when the SVD is available?

61. Derive the gradient and the Hessian of the function

$$f(x) = \sum_{i=1}^{m} h_i(x)^2 \ , \ x \in \mathbb{R}^n.$$

62. Define the *Gauss-Newton* method.

63. Define the *Levenberg-Marquardt* method.

64. Explain which cases are the most favorable for Gauss-Newton and Levenberg-Marquardt methods.

65. What is the LICQ condition?

66. Explain the definition of the sets $\mathcal{F}_x$ and $F_1(x)$ for an $x \in \Omega$?

67. State the KKT theorem and explain how it is applied.

68. State the two main lemmae used in the proof of the KKT theorem, and outline the arguments then leading to the theorem.

69. Write down the KKT-equations for the problem

$$\begin{gathered} \min\left\{x_1^2 + x_2^2 + x_3^2\right\}, \\ 1 - x_1 - x_2 - x_3 \geq 0, \\ x_2 \geq 2x_3, \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{gathered}$$

70. State conditions on the equality and inequality constraints leading to a *convex* feasible domain. Explain the KKT theorem in connection with convex problems. Can this result be applied to the problem in the previous point?

71. State the 2nd order conditions for constrained problems.

72. State the Standard form of the LP problem.

73. What are slack and surplus variables?

74. Transform the following problem to standard form:

$$\begin{gathered} \max\left(-x_1 + x_2\right), \\ x_1 \leq 2 + x_2, \\ 6 - x_2 \geq x_1 \end{gathered}$$

75. State the KKT equation for the standard form. Is a solution of the KKT equation a solution of the LP-problem?

76. What is the definition of dual problems?

77. Show that the dual problem to the Standard Problem is

$$\max b'\pi$$
$$A'\pi \leq c,$$

78. What is an extreme point of a simplex, and what is the connection to a basic point for the LP problem in standard form?

79. Explain the SIMPLEX algorithm in broad terms.

80. Define the Quadratic Programming Problem in the standard form.

81. Explain one of several ways to deal with a QP problem only involving equality constraints.

82. Explain in broad terms the idea with an active set method.

83. Explain the basic idea behind penalty and barrier methods. What happens with the Hessian for the resulting unconstrained problem?

84. Formulate a logarithmic barrier algorithm for the LP-problem in standard form.

85. What does the notation $C[a,b]$ and $C^1[a,b]$ mean?

86. Define a functional and the Gâteaux derivative.

87. What is the Gâteaux derivative of a linear functional?

88. State a simple way to compute the derivative.

89. State simple sufficient conditions for the validity of

$$\frac{d}{dt}\int_a^b h(x,t)\,dx = \int_a^b \frac{\partial h(x,t)}{\partial t}dx.$$

90. What is $\delta J(y;v)$ for $y \in C^1[a,b]$ when

$$J(y) = \sin y'(a) + \cos y(b)?$$

91. Let

$$J(u) = \frac{1}{2}\int_D \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right\} dxdy$$

for a nice function u defined on a nice domain in $\mathbb{R}^2$. Prove that

$$\delta J(u;v) = \int_D \left\{ \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right\} dxdy.$$

92. Define a convex and a strictly convex functional.

93. Consider a convex functional defined on a convex set $\mathcal{D}$ of functions. Prove that a function $y_0 \in \mathcal{D}$ where

$$\delta J(y_0;v) = 0$$

for all feasible directions is a minimum of J on $\mathcal{D}$.

94. Define the standard functional.
95. In connection with the standard functional, what does it mean that the function $f(x, y, y)$ is partially convex, strongly convex and strictly convex?
96. How does the previous point connect to the convexity of the standard functional?
97. State the derivative of the standard functional. Carry out a partial integration and state the Euler equation and the 3 common cases related to the boundary conditions.
98. State how one may be able to solve the constrained problem

$$\min_{y\in\mathcal{D}} J(y),$$
$$G_i(y) = a_i, i = 1, \cdots, N$$

by forming the Lagrangian $\mathcal{L}$ and solving $\delta\mathcal{L} = 0$.