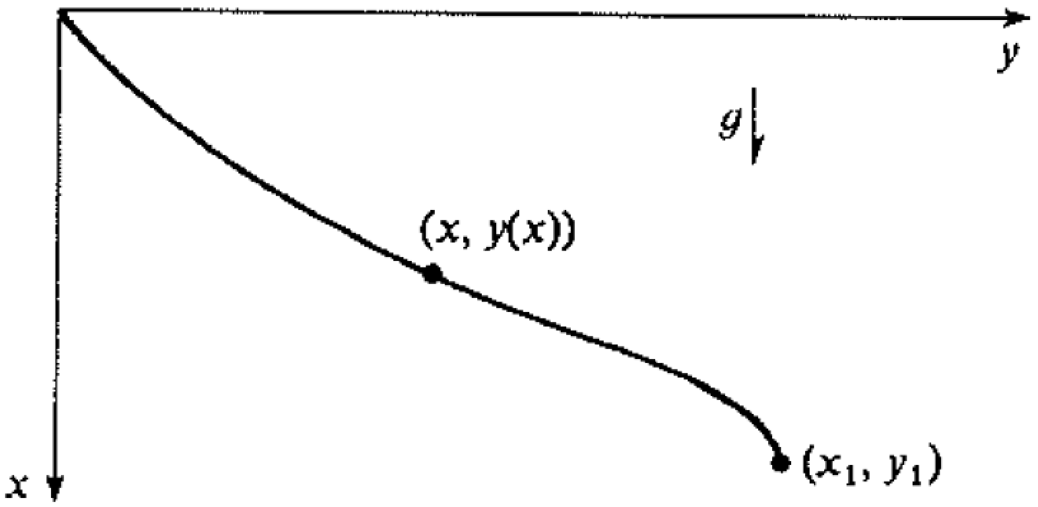
THE BRACHISTOCHRONE – A PARTIAL SOLUTION
(Troutman p. 66 – 68)



Kinetic energy = lost potential energy:

$$mgx = \frac{1}{2} m v^2 (x)$$

Path:

$$ds = \sqrt{1 + y'(x)^2}\, dx$$

Travel time:

$$T(y) = \int_0^{x_1} \frac{ds}{v} = \int_0^{x_1} \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gx}}\, dx$$

Problem: The opt. path is not necessarily monotone in x.

$$f(x,y') = \frac{\sqrt{1+y'(x)^2}}{\sqrt{2gx}}$$

$\sqrt{1+y'(x)^2}$ is strongly convex

$$\sqrt{2gx} > 0$$

$\Rightarrow$ $T(y)$ is strictly convex

Euler equation: $$\frac{d}{dx}\left[f_{y'}(x,y') \right] = 0$$

$$\frac{y'(x)}{\sqrt{2gx}\sqrt{1+y'(x)^2}} = \frac{1}{\tilde{c}} \quad \Rightarrow \quad \frac{y'(x)}{\sqrt{1+y'(x)^2}} = \frac{\sqrt{x}}{c}$$

Equation is only meaningful for $y'(x) > 0$

and then $c > \sqrt{x}$

Square equation: $$\frac{y'(x)^2}{1+y'(x)^2} = \frac{x}{c^2}$$

or

$$y'(x)^2 = \frac{x}{c^2 - x}$$

Implicit solution:

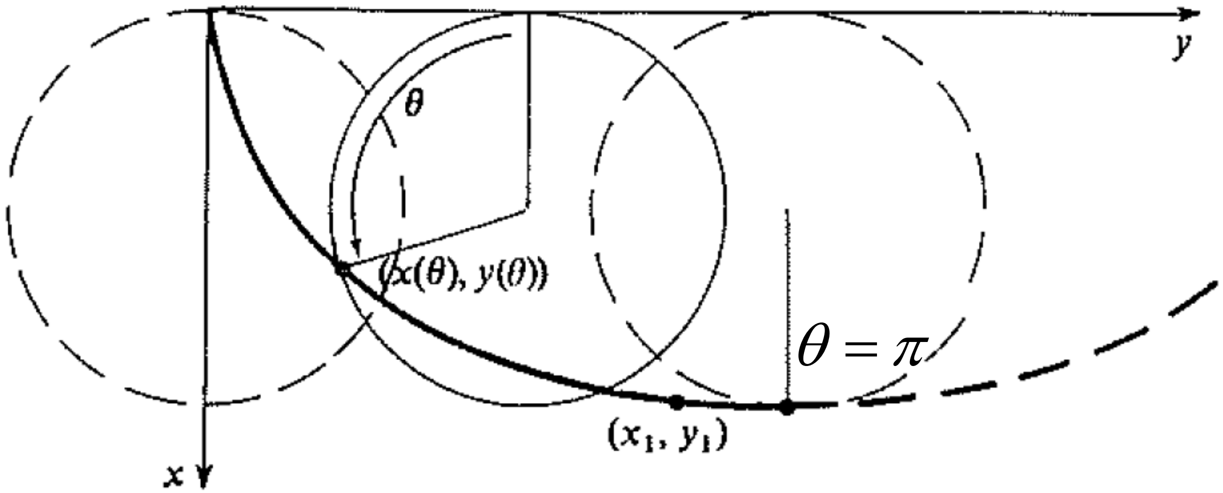
$$y(x) = \int_0^x \sqrt{\frac{t}{c^2 - t}}\,dt$$

Set $t = \frac{c^2}{2}(1 - \cos\theta)$

By a small trick using trig. substitutions,

$$x(\theta) = c^2(1 - \cos\theta)$$
$$y(\theta) = c^2(\theta - \sin\theta)$$

$$\left(c := \sqrt{2}c\right)$$



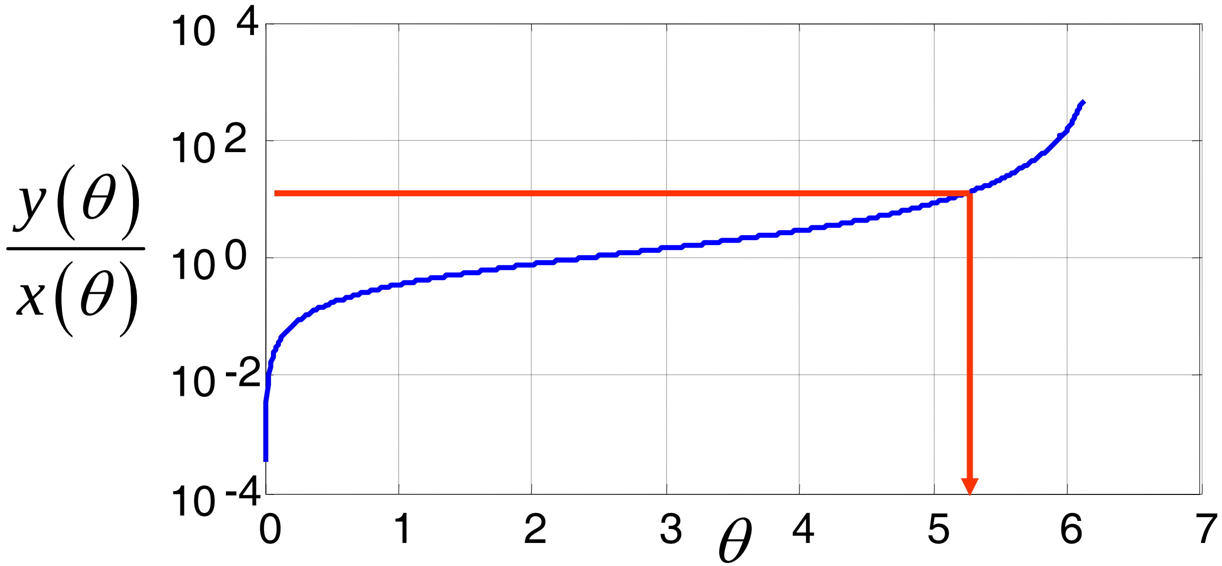
Cycloid

There is always a (unique) cycloid of the form

$$x(\theta) = c^2 (1 - \cos\theta)$$

$$y(\theta) = c^2 (\theta - \sin\theta)$$

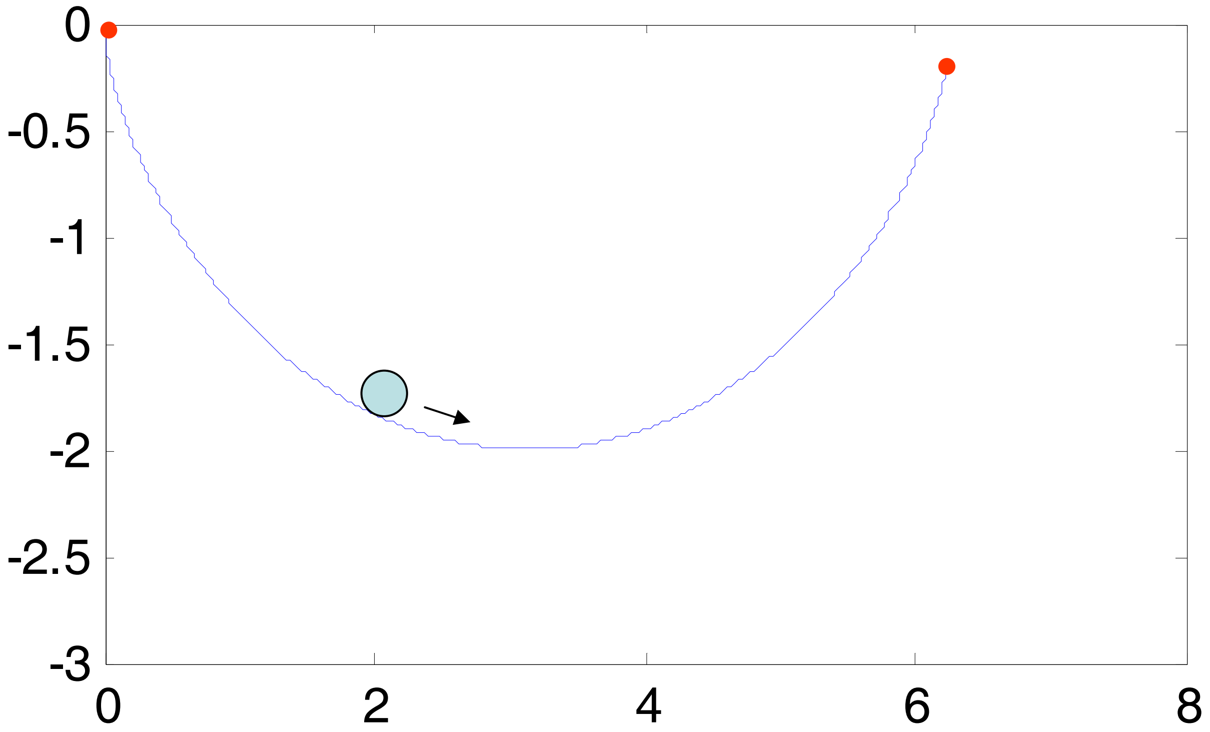
for all $(x_1, y_1),\, x_1, y_1 > 0$:



However, it is possible to write $y = y(x)$ only for $\theta \leq \pi$

(But that solves our problem in that case, since then T(y) is strictly <u>convex</u>)

The solution is a cycloid in general!



(Proved later in Troutman)