TMA 4180 Optimeringsteori

INVERSE PROBLEMS AND OPTIMIZATION

Harald E. Krogstad, IMF, Spring. 2010

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- Inverse problems
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- Examples
- Techniques

Jeopardy - The everyday Inverse problem:

"It was in 1905"

"When did Einstein publish his Theory of Relativity ?"

"When did Robert Koch get the Nobel Price in Medicine?"

"When was my grandmother born?"

"When did Norway and Sweden split up from the union?"

- The information is incomplete and non-conclusive

- Different input leads to the same result

- The ”most probable” input depends on circumstances

direct problem: given the question, find the answer

inverse problem: given the answer, find the question

Well-posed problems (Hadamard, 1923):

- a solution always exists
- there is only one solution
- a small change in the problem leads to a small change in the solution

Ill-posed problems:

- a solution may not exist
- there may be more than one solution
- a small change in the problem leads to a big change in the solution

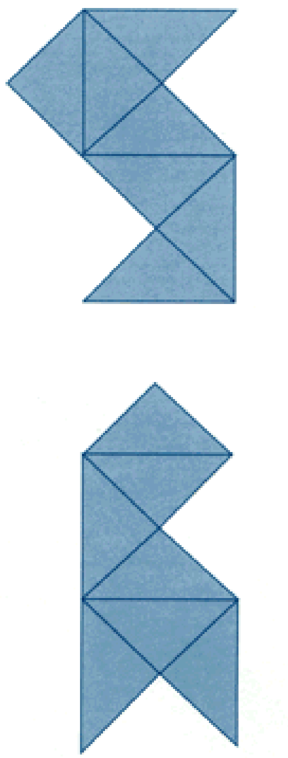
INVERSE PROBLEMS ARE (ALMOST) ALWAYS ILL-POSED!

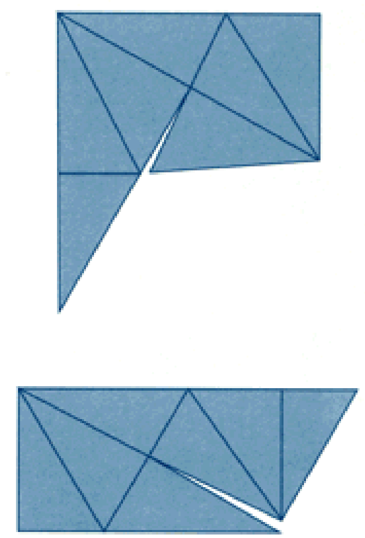
SOME FAMOUS INVERSE PROBLEMS

- Marc Kac (1966): "Can you hear the shape of a drum?"
- Computer Tomography
- Seismic Inversion
- Image Restoration

NO, you can't hear the shape of a drum, but you can hear

- the area,
- the length of the boundary,
- and the number of holes!



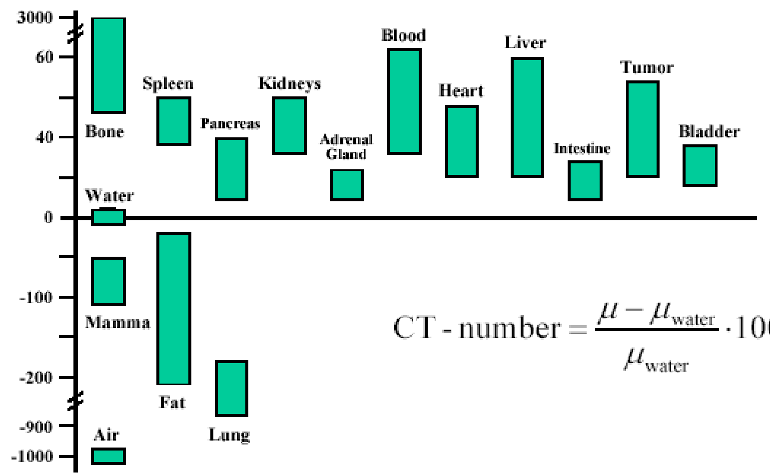


http://www.ams.org/new-in-math/hap-drum/hap-drum.html

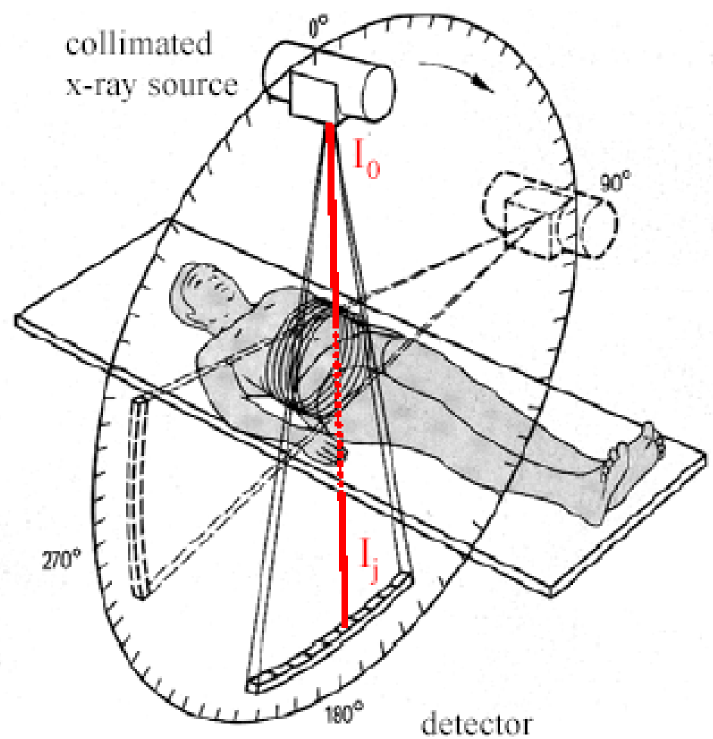
COMPUTER TOMOGRAPHY

http://www.iwr.uni-heidelberg.de/groups/ngg/Tutorial/TutCT_121203_Lauritsch.pdf

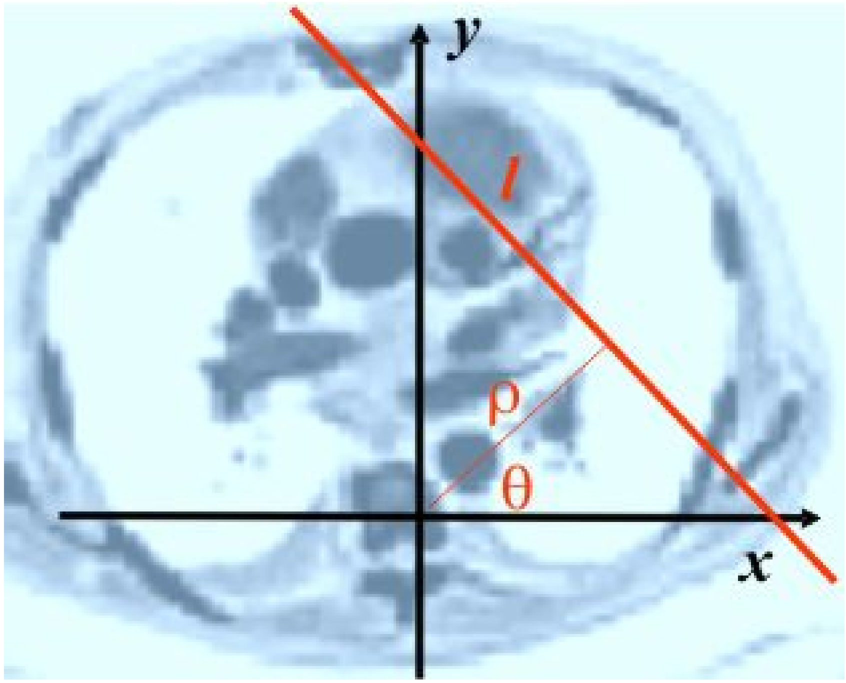
CT-numbers of tissue in Hounsfield units (HU)



$$\text{CT-number} = \frac{\mu - \mu_{\text{water}}}{\mu_{\text{water}}} \cdot 1000$$



$$I_j = I_0 \exp\left(-\int_{line_j} \mu(\mathbf{x}) dl \right)$$

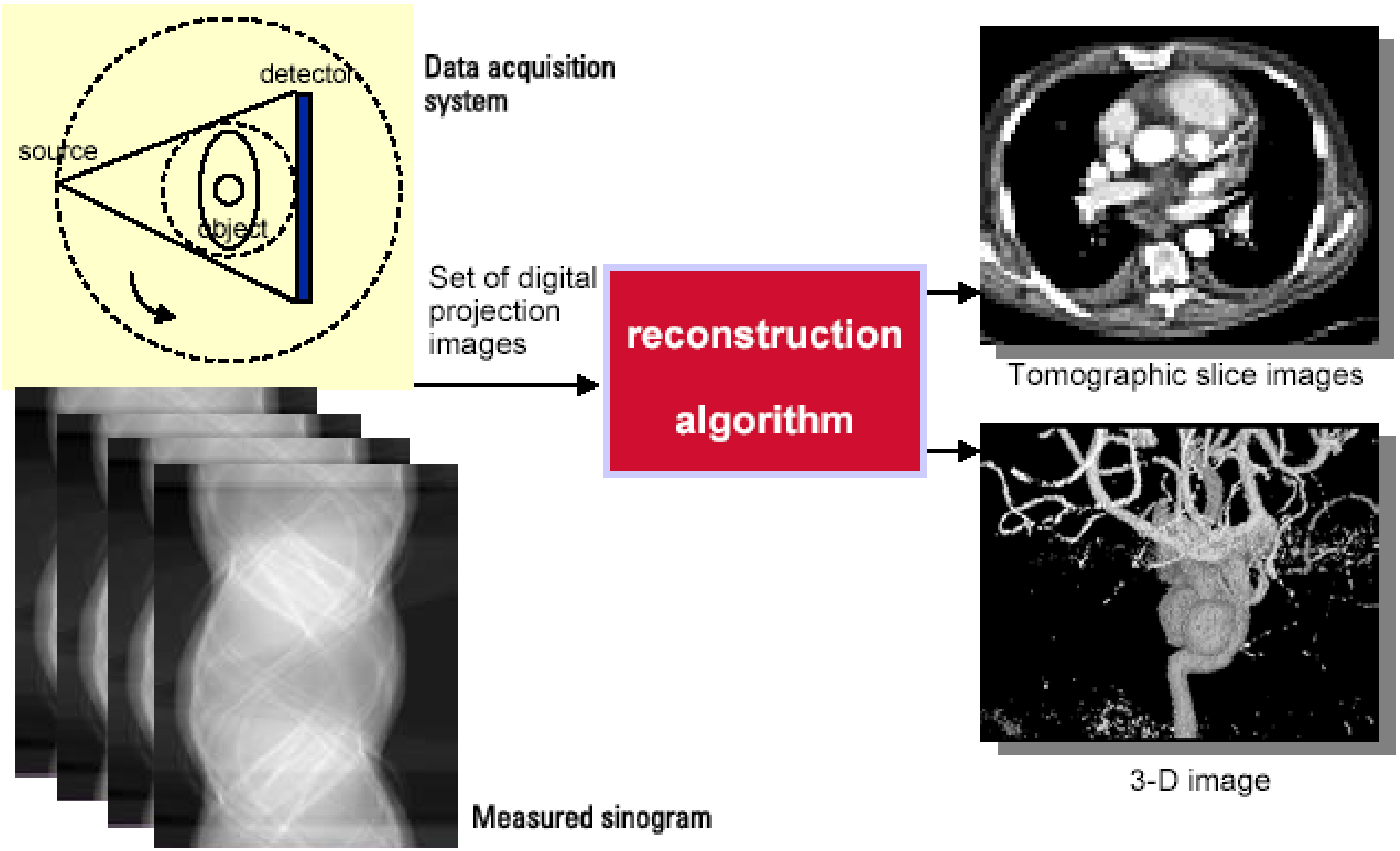


Radon transform: Based on line integrals

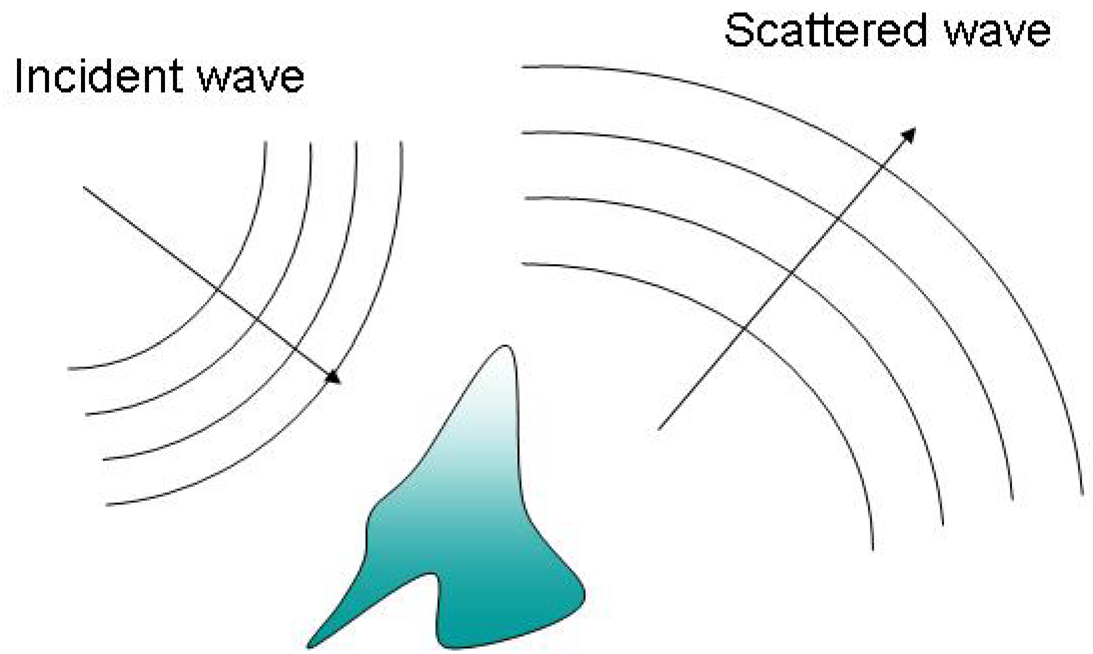
$$\Re\mu(\rho,\theta)=\hat{\mu}(\rho,\theta)=\int_{-\infty}^{\infty}\mu(\rho\cos\theta-t\sin\theta,\rho\sin\theta+t\cos\theta)\,dt$$

Inverse problem: $\mu(\mathbf{x})=\Re^{-1}(\hat{\mu}(\rho,\theta))$

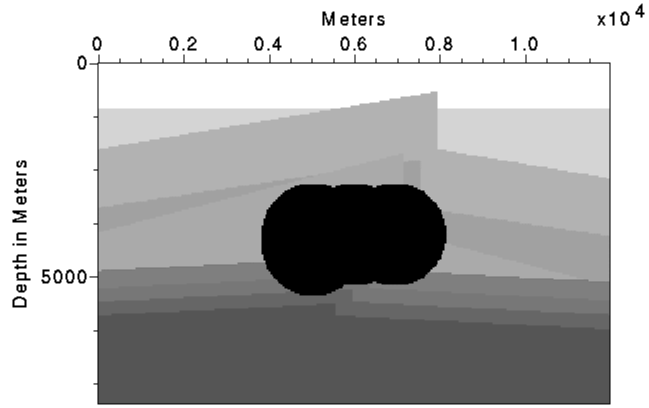
Reconstructive Methods



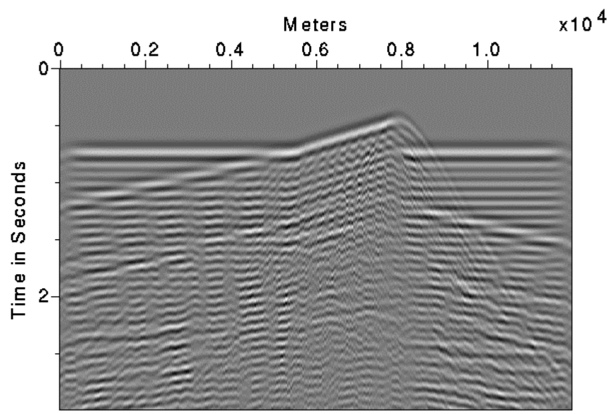
INVERSE SCATTERING



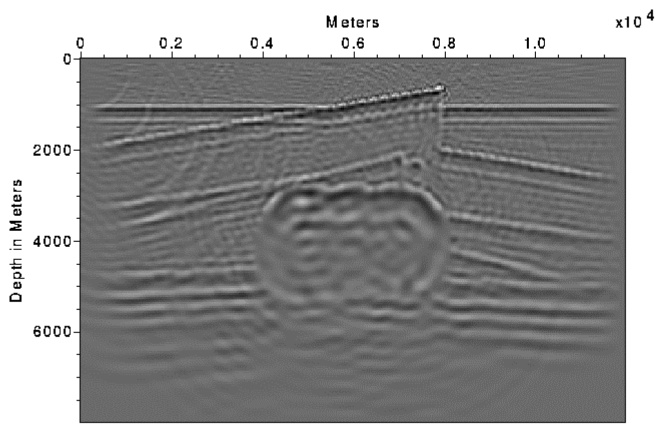
Seismic inversion



Salt Intrusion Model, 200 by 300 grid



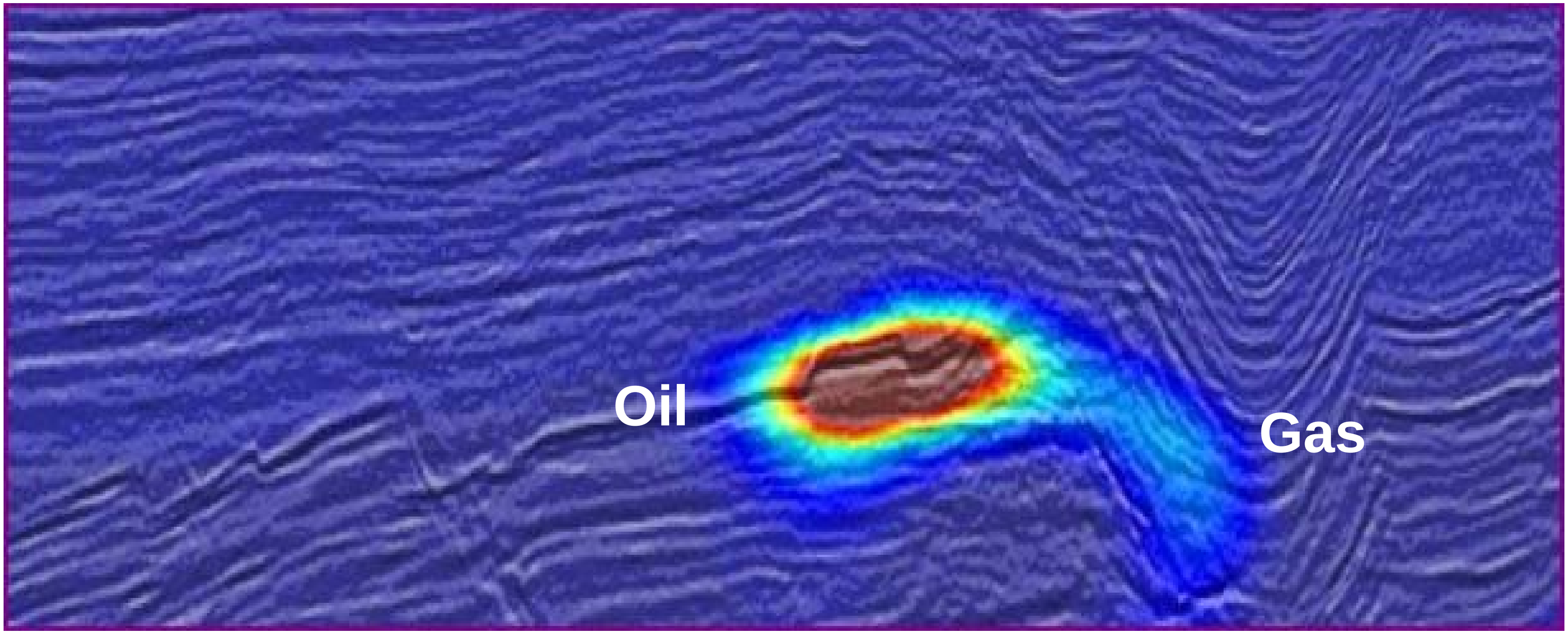
Salt Intrusion Exploding Reflector Data, dt=0.002



Salt Intrusion Reverse Time Migration, Grid Spacing = 40 Meters

http://www.mgnet.org/~douglas/Classes/cs521-s00/asdf/asdf.ppt

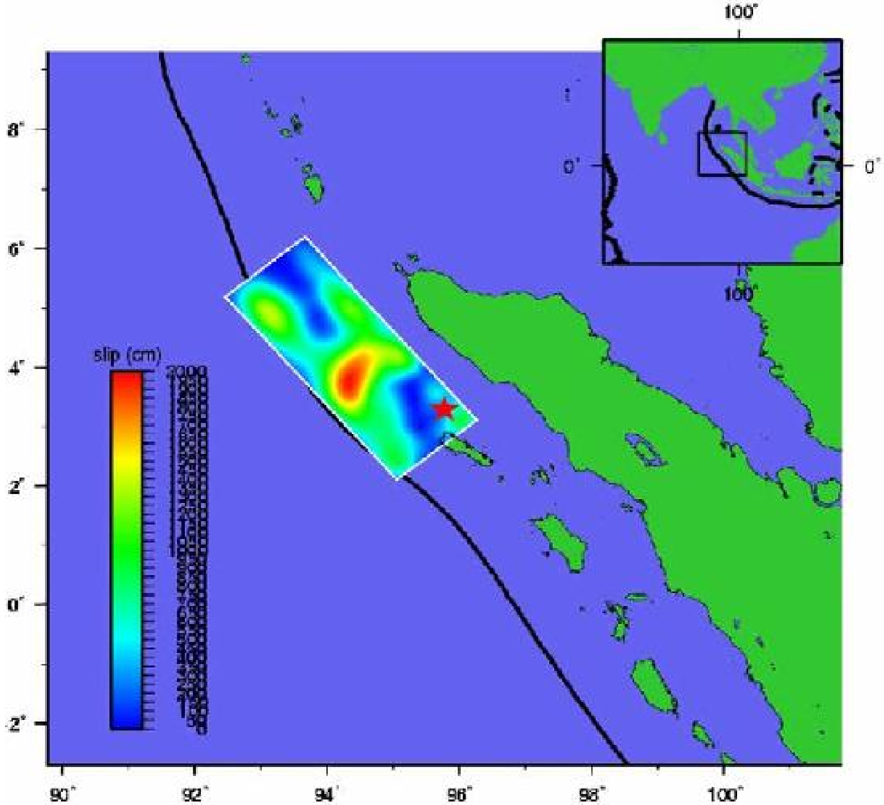
Resistivity map for a major North Sea field, 3D EM method. (Data courtesy of EMGS and Statoil Hydro)



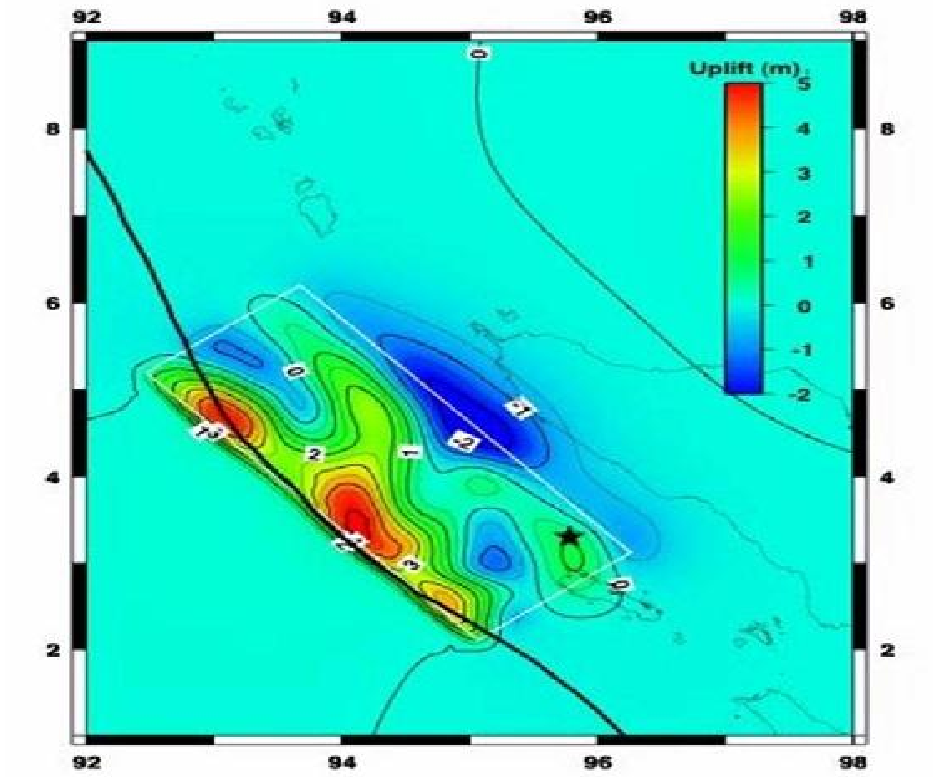
(Houston Geological Society)

Compute sea floor motion from seismic recordings:

Slip



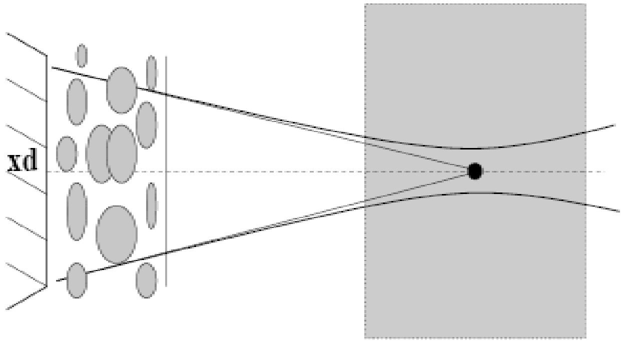
Uplift

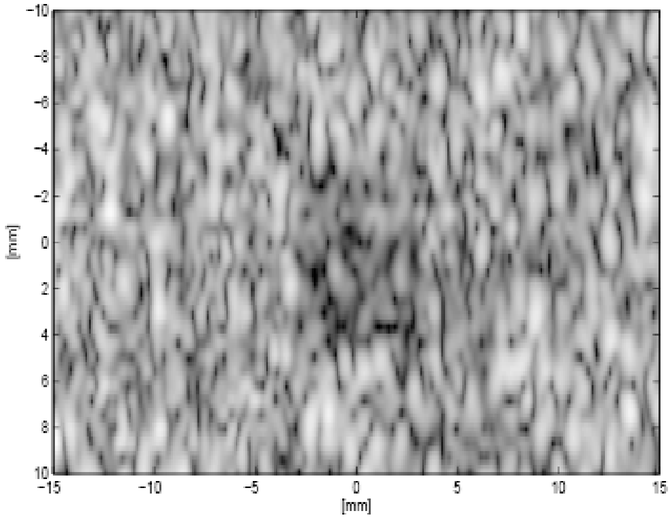


(The December 26, 2004, Sumatran Tsunami)

(Bjørn Gjevik, UiO. Published by Caltech)

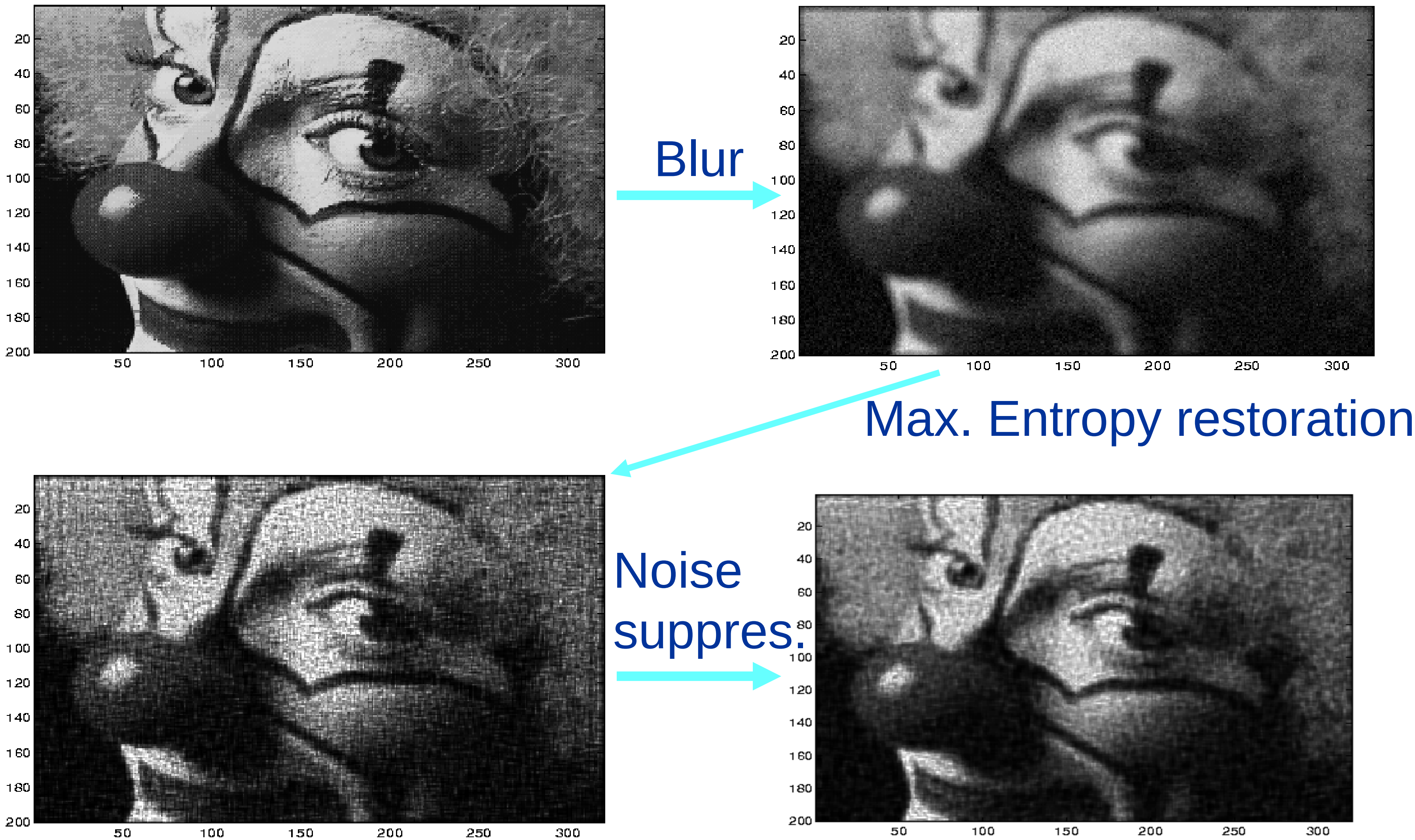
MEDICAL ULTRASOUND



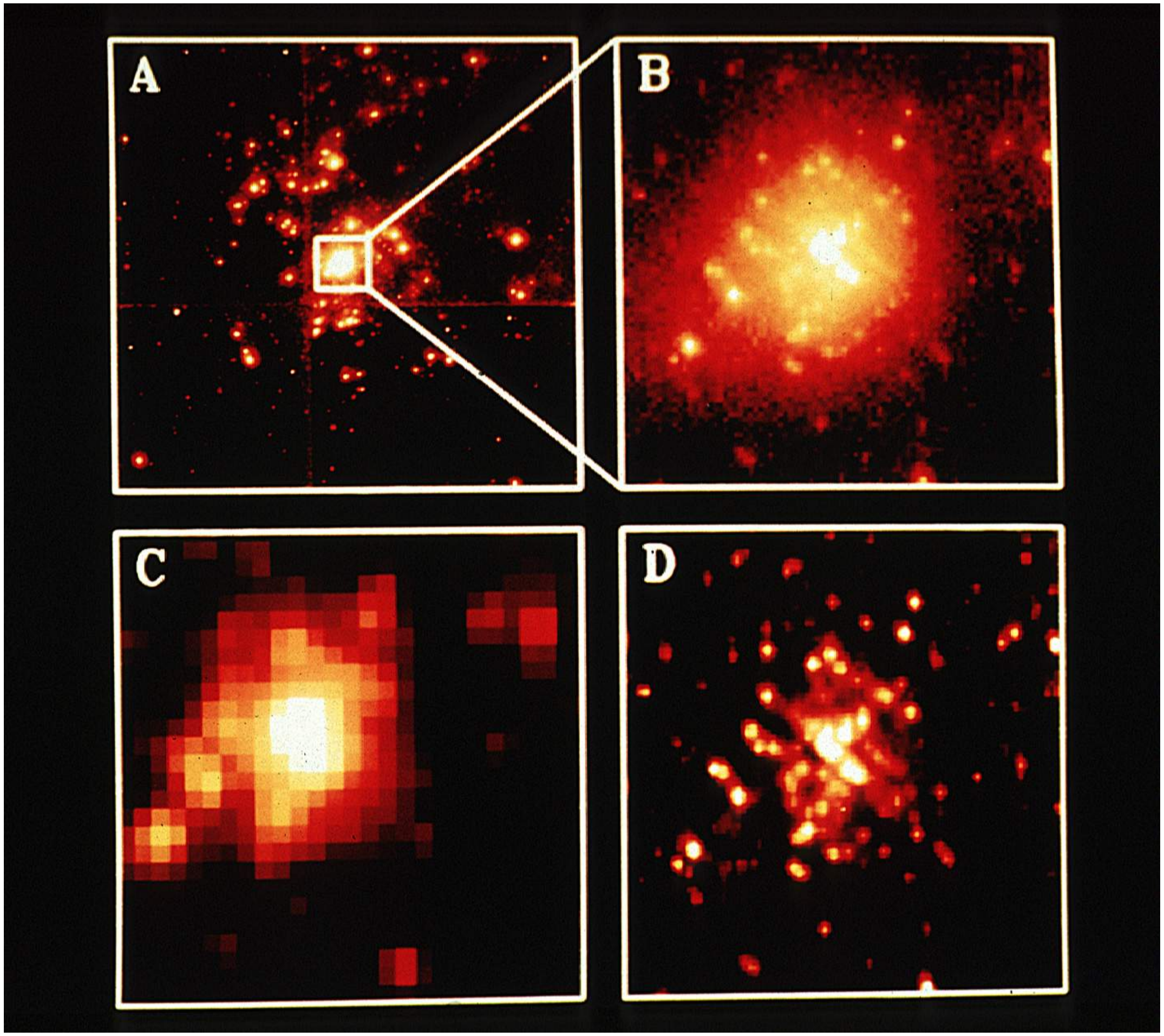


(Trond Varslot)

IMAGE RESTORATION



http://mip.ups-tlse.fr/~noll/noll_tutorial.html



A: Original HST photo. B: Enlarged section.
C: Ground telescope image. D: Digitally enhanced image.

LINEAR, NOISE-FREE PROBLEMS

Input	Filter	Observation
$x \in \mathbf{R}^n$	$\longrightarrow \boxed{A} \longrightarrow$	$b = Ax \in \mathbf{R}^m$

Singular value decomposition: $A = U\Sigma V'$

$$x^* = A^+ b = V\Sigma U' b = \sum_{j=1}^{rank(A)} v_j \frac{u_j' b}{\sigma_j}$$ (unstable)

Regularization: *Truncated SVD (TSVD)*:

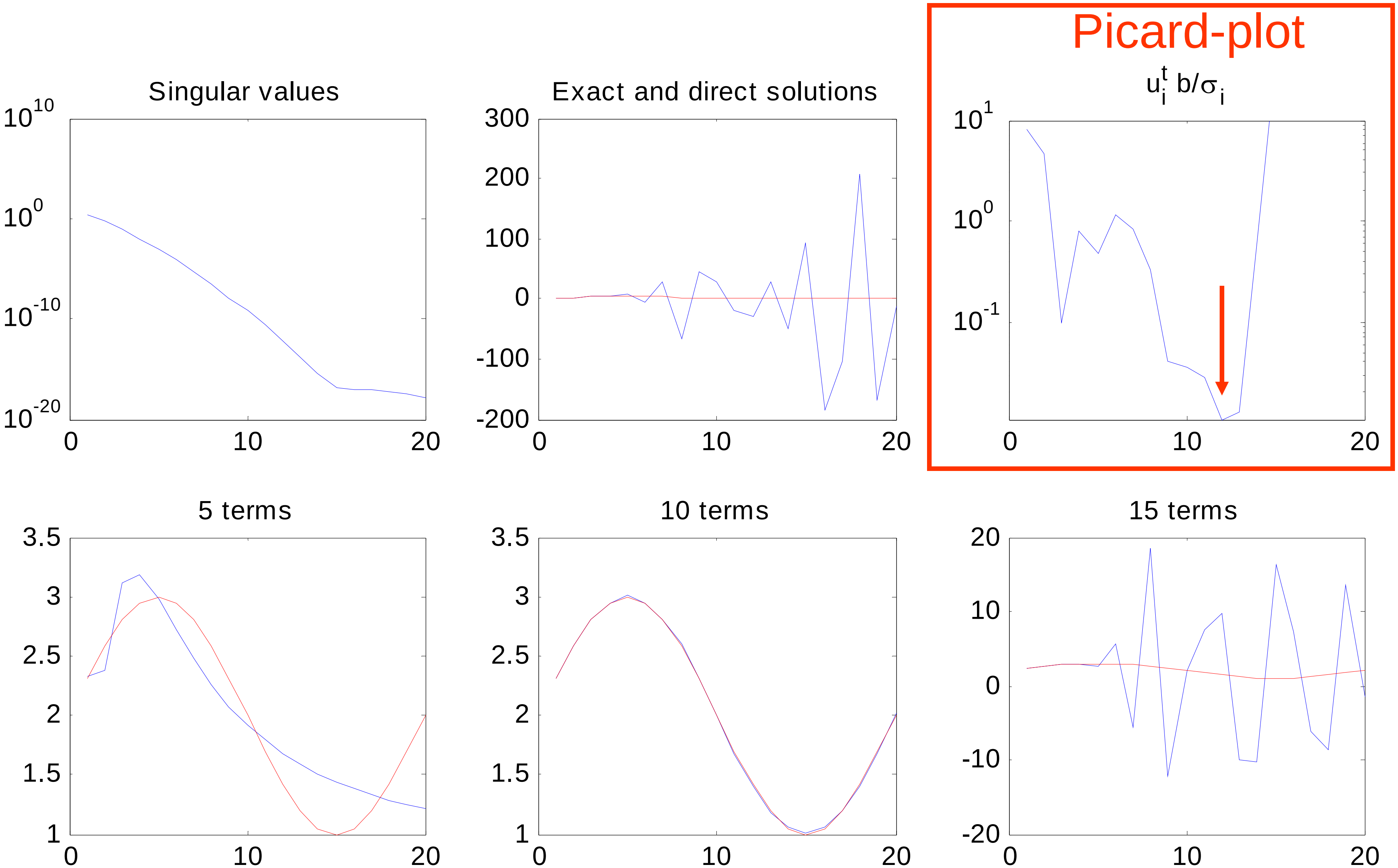
$$x^K = \sum_{j=1}^{K} v_j \frac{u_j' b}{\sigma_j}$$

Example: The Hilbert Matrix System

$$Hx = b$$

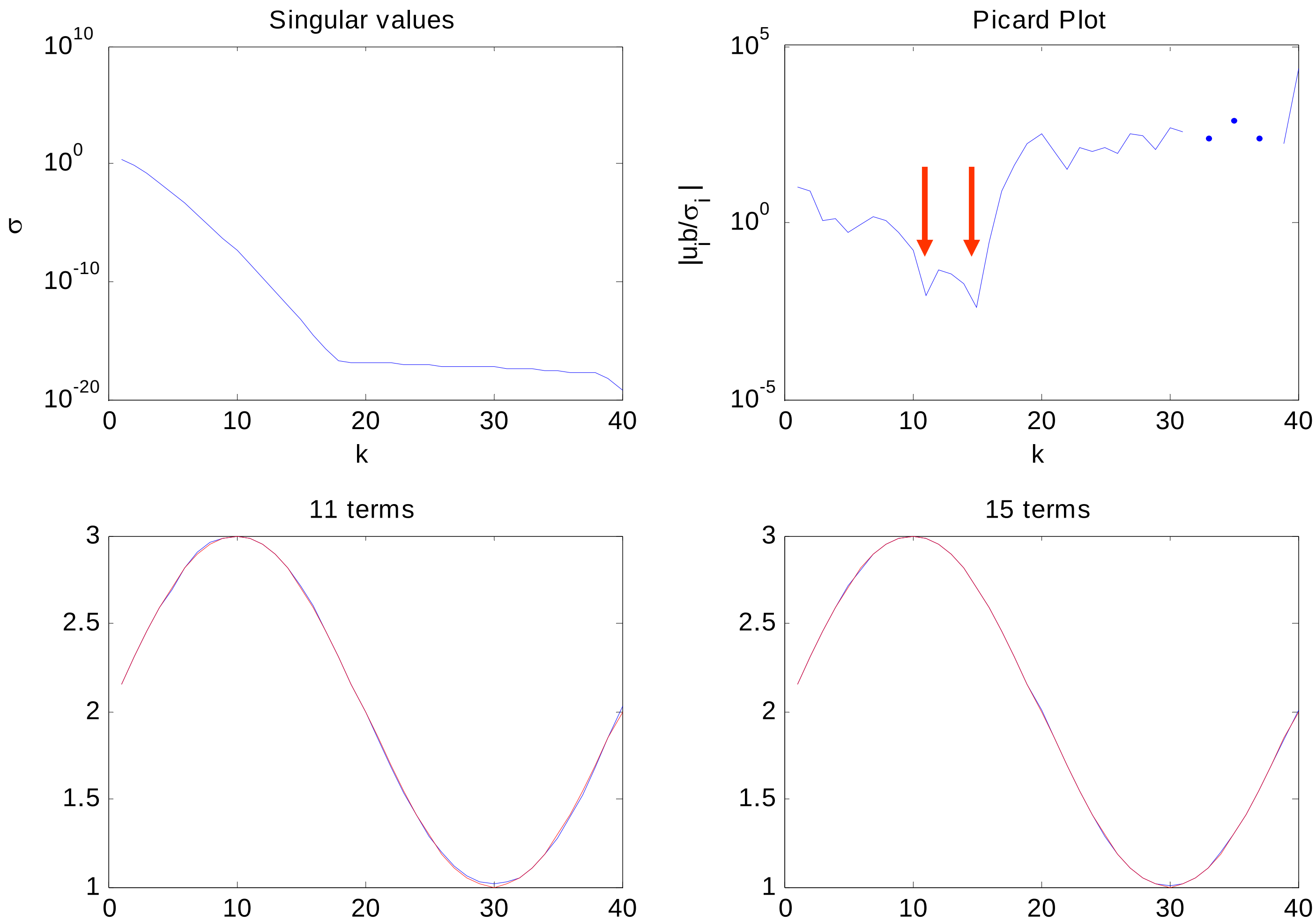
$$H_n = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & & \cdots & \frac{1}{n+1} \\ \vdots & & & & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

$$h_{ij} = \int_0^1 x^{i-1} x^{j-1} dx$$



(cond. Number = 1.5×10^{18})

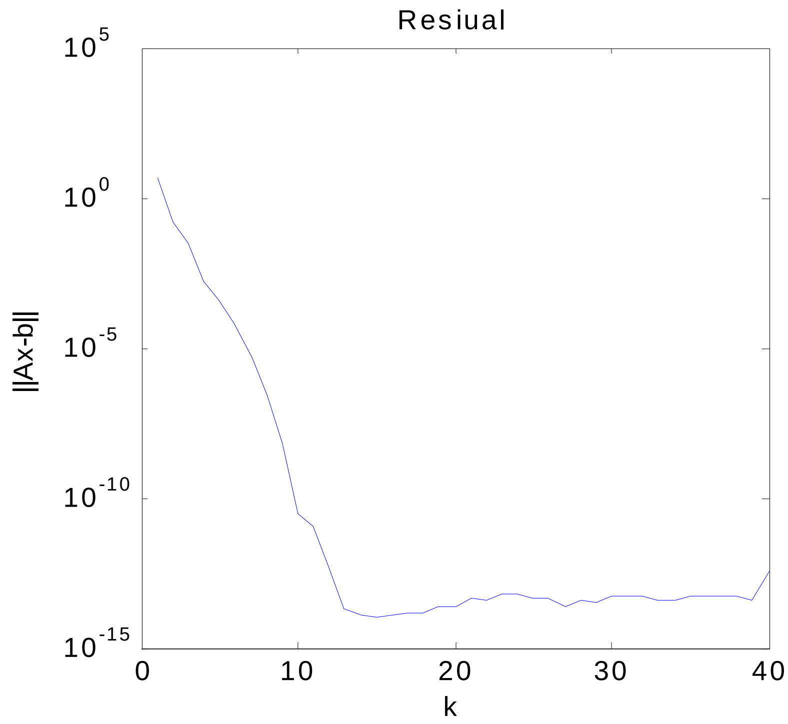
Hilbert problem cont. ...

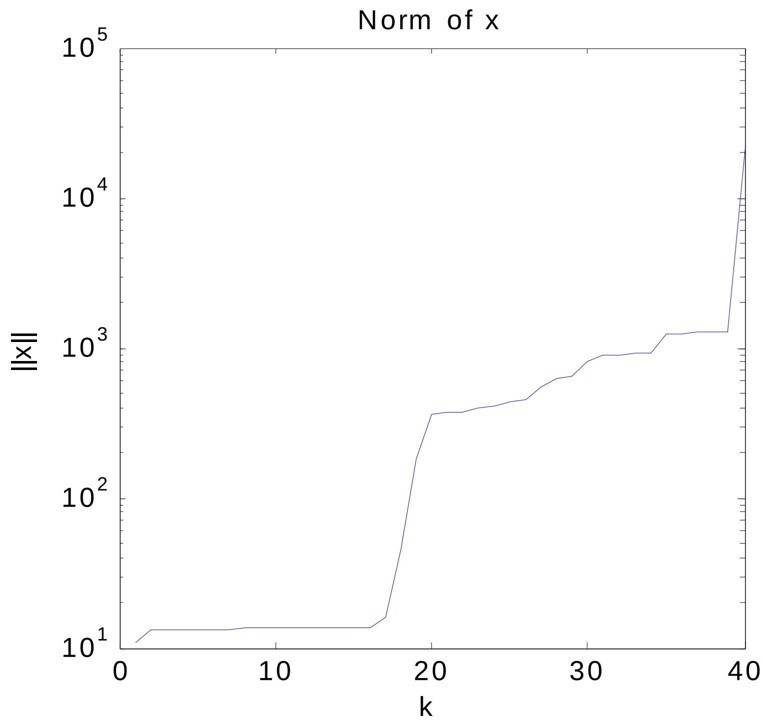


The Picard plot shows $|\alpha_j|$: $x^* = \sum_{j=1}^{n} \alpha_j v_j, \quad \alpha_j = \dfrac{u_j{}'b}{\sigma_j}$

Hilbert problem cont. ...

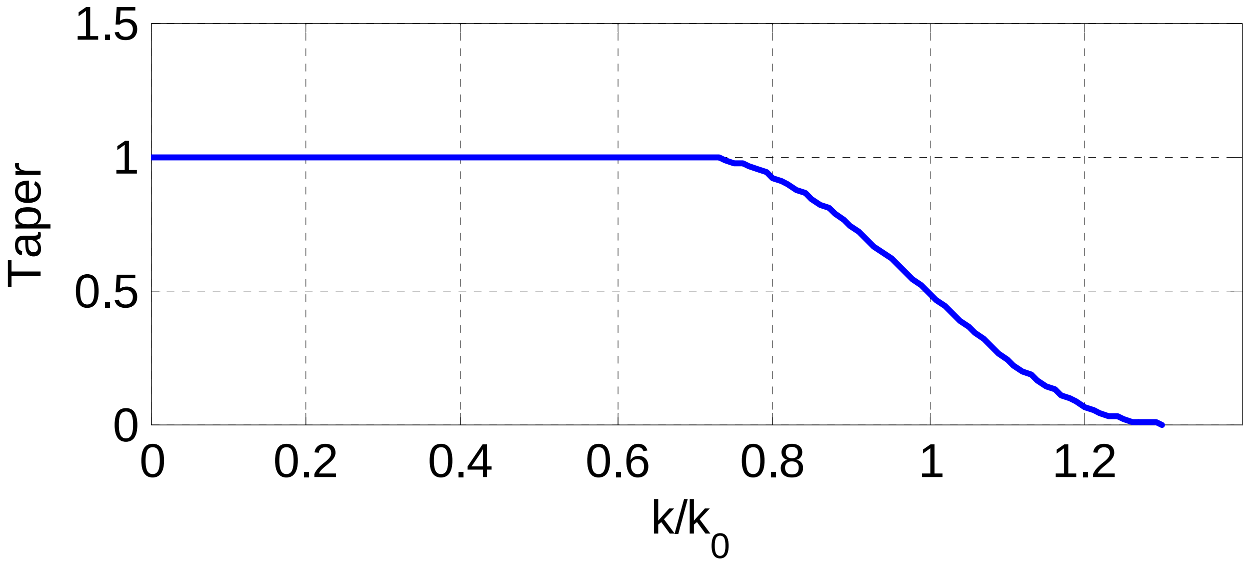
WARNING:
A small residual does not mean a good solution for ill-conditioned problems:





Hilbert problem cont. ...

A smooth cut-off: The *cosine taper*



TIKHONOV REGULARIZATION

- We expect that X is close to X_0 :

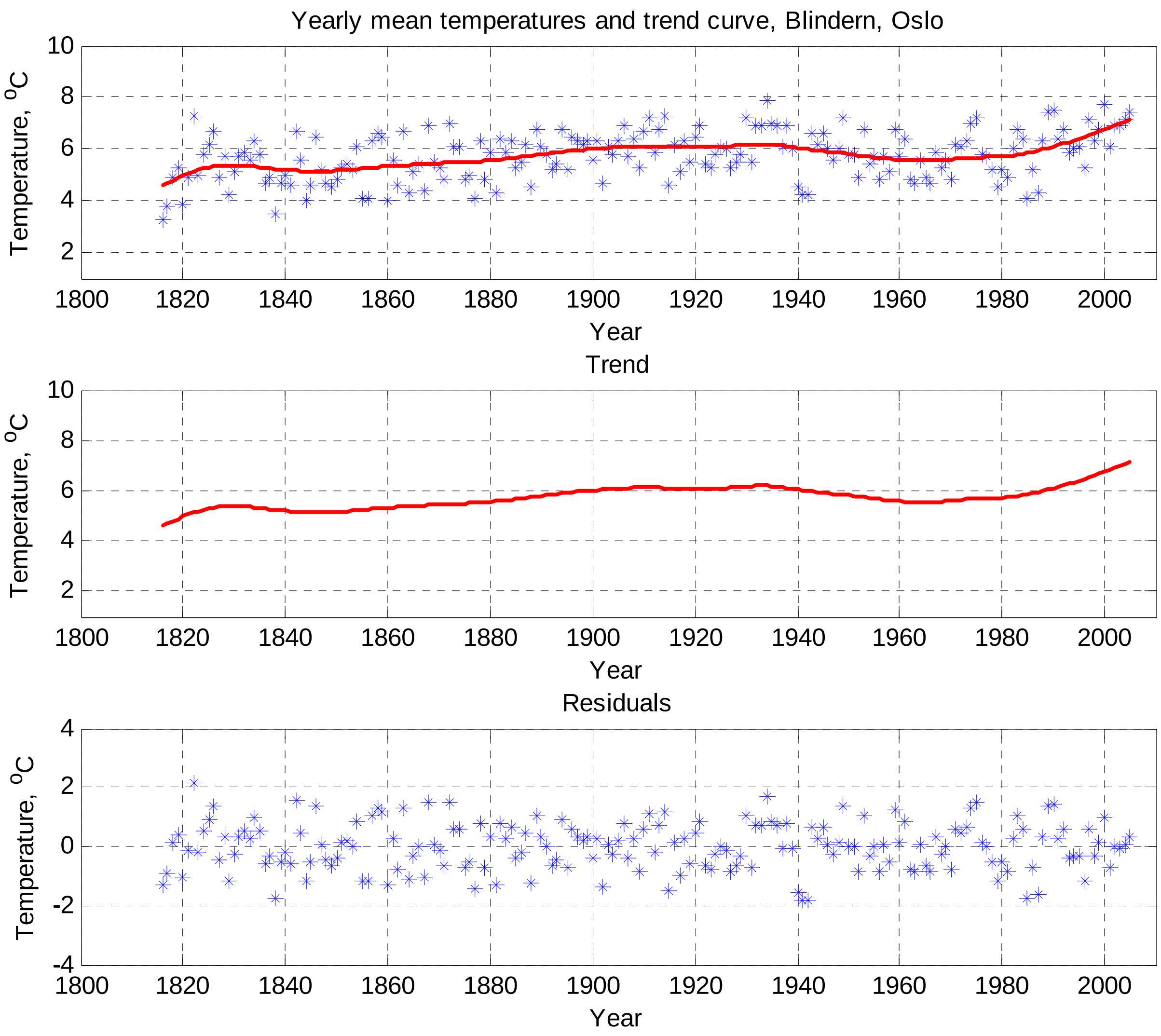
$$X^* = \arg\min_X \left\{ \underbrace{\left\| T(X) - Y \right\|^2}_{\text{Error term}} + \overset{\text{Weight}}{\mu} \underbrace{\left\| X - X_0 \right\|^2}_{\text{Regularization term}} \right\}$$

- We expect that X is *smooth*:

$$X^* = \arg\min_X \left\{ \left\| T(X) - Y \right\|^2 + \mu \left\| L(X) \right\|^2 \right\}$$

(The *L* operator punishes irregularities)

Hodrick-Prescott Filter



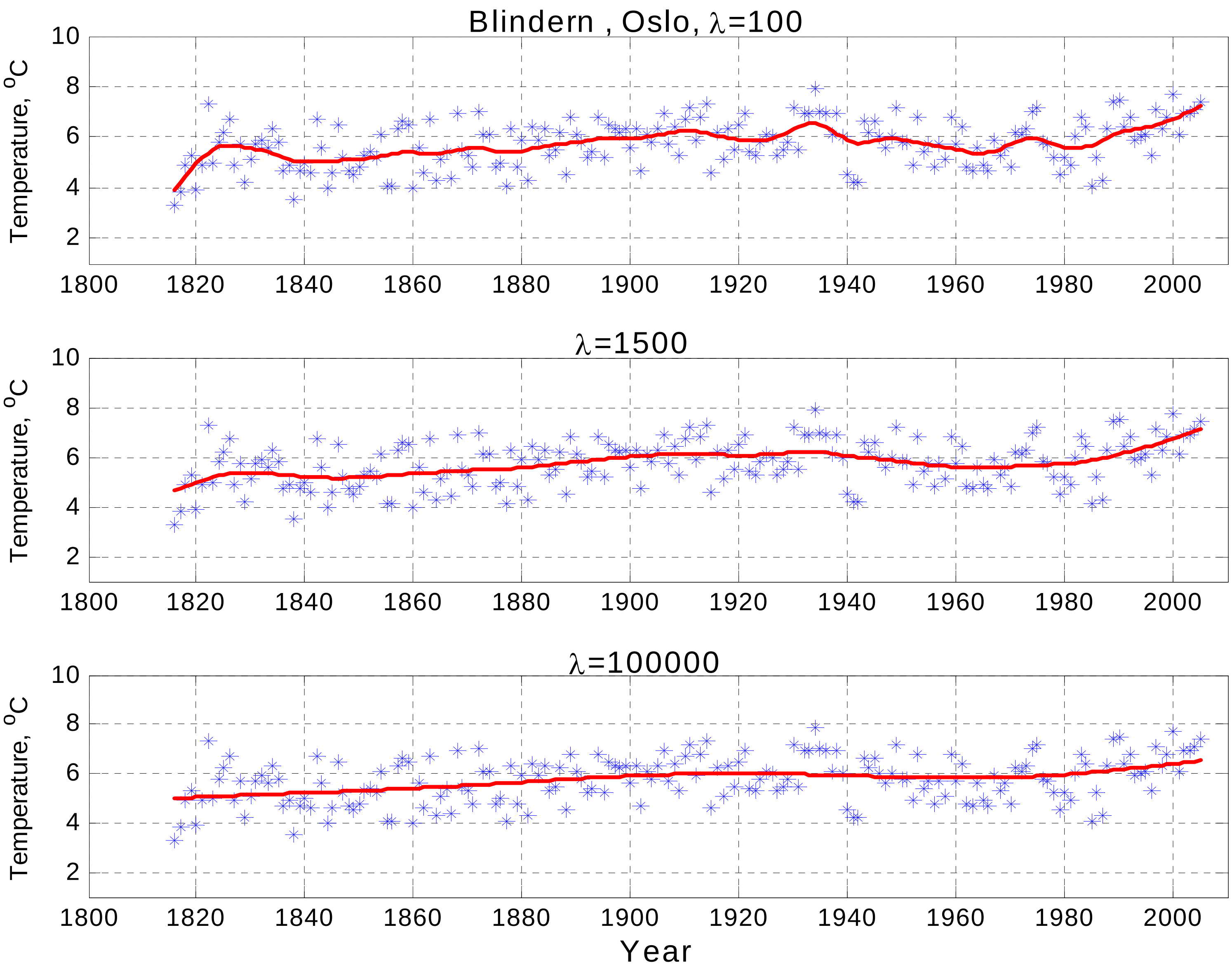
$\{x_i\}$ Data

$\{t_i\}$ Trend curve

$$\min_t \left(\sum_{i=1}^{N} (x_i - t_i)^2 + \lambda \sum_{i=2}^{N-1} (t_{i+1} - 2t_i + t_{i-1})^2 \right)$$

Error term　　　　Regularization

N unknowns!


Blindern , Oslo, λ=100
λ=1500
λ=100000
Temperature, °C
Year
10
8
6
4
2
1800
1820
1840
1860
1880
1900
1920
1940
1960
1980
2000

THE KEY PROBLEM

- How to select the best/most probable/optimal solution

  - How to dampen ”ill-conditionness”
    (regularize - but not exaggerate!)

Example: *Tikhonov regularization*

$$X^* = \arg\min_X \left\{ \left\| T(X) - Y \right\|^2 + \mu \left\| X - X_0 \right\|^2 \right\}$$

μ **small**: solution unstable (ill-conditioned problem)

μ **large**: $X^* \approx X_0$ (even if our ”knowledge” X_0 is false!)

FINDING THE OPTIMAL REGULARIZATION:

- Truncated SVD – Picard Plot
- Wiener Filter
- Morozov's Discrepancy Principle
- The L-curve
- Iteration truncation

There is no generally best method!

FREDHOLM INTEGRAL EQUATIONS

$$\boxed{y(t) = \int_0^1 K(t,\tau)\,x(\tau)\,d\tau}$$

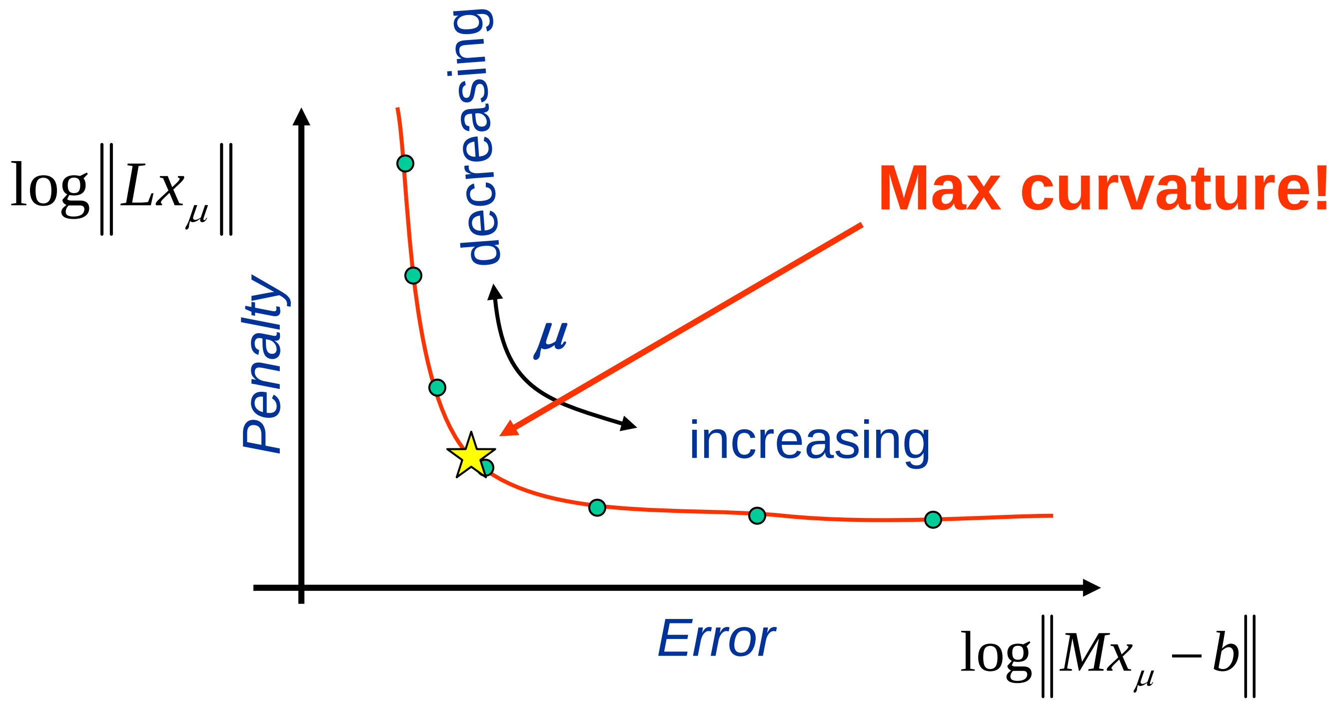
$K(t,\tau)$ is called the *kernel*

Discretized Fredholm Integral Equations are generally ill-conditioned

L-CURVE ANALYSIS

(Per Chr. Hansen, DTU)

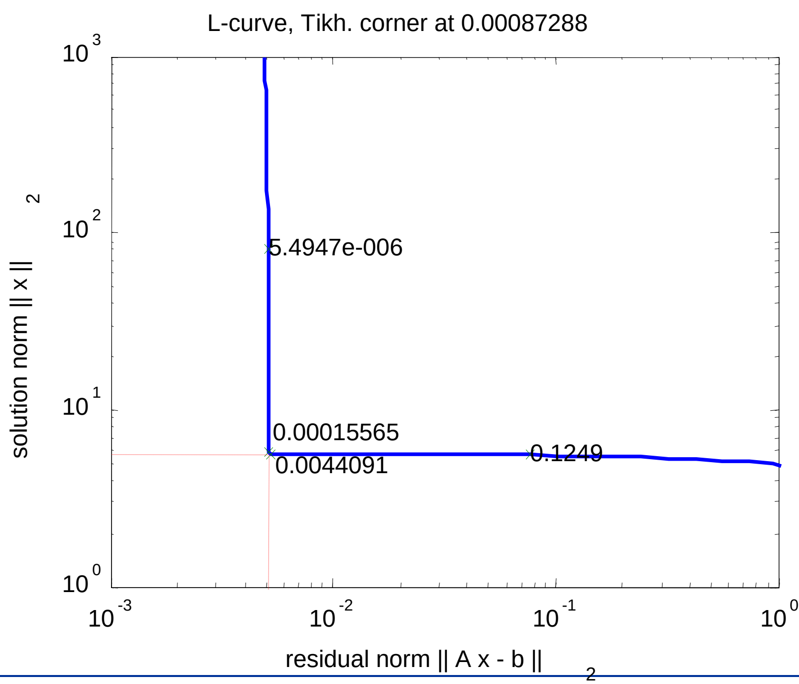
Plot the two terms in the Tikhonov objective function:

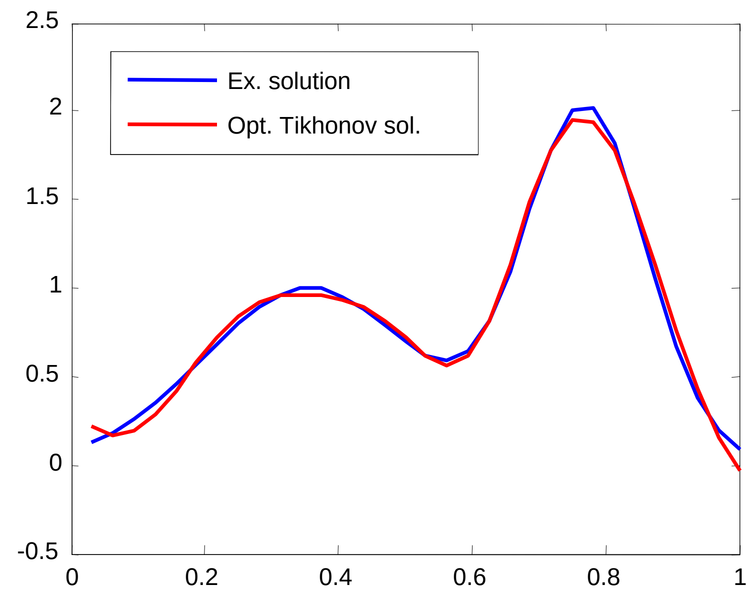


$$\int_{-\pi/2}^{\pi/2} K(t,s)\,x(s)\,ds = b(t)$$

$$K(t,s) = (\cos s + \cos t)\frac{\sin^2\left(\pi(\sin s + \sin t)\right)}{\pi^2(\sin s + \sin t)^2}$$

$$x_{sol}(t) = a_1 \exp\left(-\frac{(t-t_1)^2}{c_1^{\ 2}}\right) + a_2 \exp\left(-\frac{(t-t_2)^2}{c_2^{\ 2}}\right)$$





ITERATIVE METHODS

The basic iterative method for a linear equation

$$Ax = b,\ A > 0,$$

is

$$x_{k+1} = x_k + \omega\left(b - Ax_k\right),\ k = 0,1,\cdots$$

("fix-point iteration")

ω is called a *relaxation parameter*.

Convergence if ω is chosen so that

$$\left\| I - \omega A \right\| < 1$$

Fix point iteration is used for inverse problems as a *general* technique where the

number of iterations is the regularization!

Input | Nonlinear filter | Observation

$$x \in \mathbf{R}^n \longrightarrow \boxed{M} \longrightarrow b = M(x) \in \mathbf{R}^m$$

$$\boxed{x_{k+1} = x_k + \omega\big(b - M(x_k)\big),\ k = 0, 1, \cdots, k_{opt}}$$

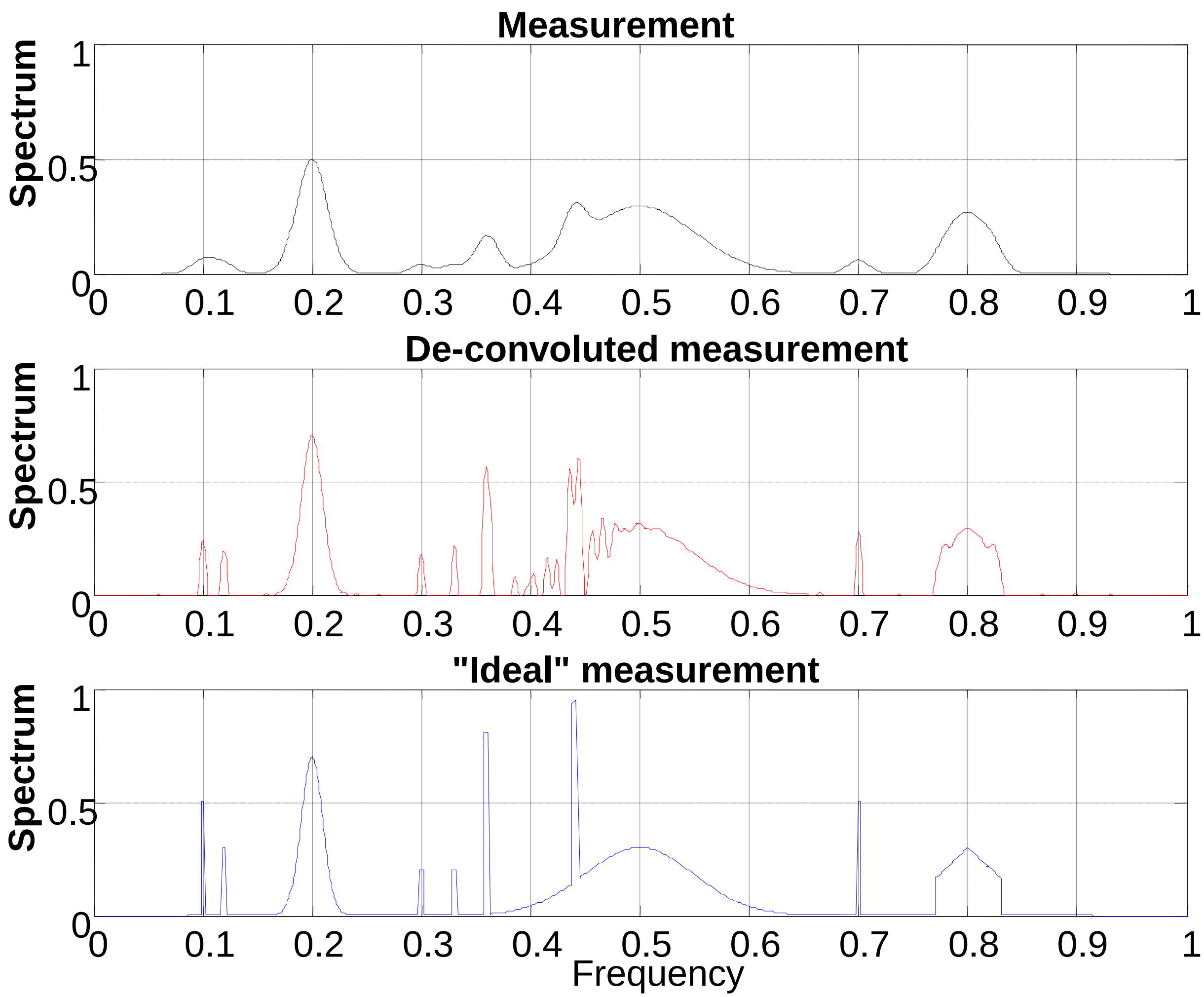
Common name: ***vanCittert iteration***

DE-CONVOLUTION OF SPECTRAL DATA

- Astronomy
- Mass spectrography
- Optics
- Nuclear Magnetic Resonance

Measurement should consist of narrow “spectral lines”,
but the instrument “blurs” the lines:

“De-blurring” is needed!



$$x_{k+1} = \max\left[0, x_k + \omega\left(b - g_{blur} * x_k\right)\right], k = 0, 1, \cdots, k_{opt}$$

2000 iterations!

A very common situation in image processing:

$$BI(\mathbf{x}) = f_{PS} * I(\mathbf{x}) = \int_{\mathbf{R}^2} f_{PS}(\mathbf{x}-\mathbf{y}) I(\mathbf{y})\, d\mathbf{y}$$

I	: Image
BI	: Blurred image
f_{PS}	: Point Spread Function

$BI \longrightarrow I$: *"Deconvolution"*

$$I_{k+1} = \max\left[0, I_k + \omega\left(BI - f_{PS} * I_k\right)\right], \quad k = 1, 2, \ldots, k_{stop}$$

(NB: Should have some idea about f_{PS} !


Blurred image
ω=2, It.no.=4
ω=2, It.no.=8
ω=2, It.no.=12
ω=2, It.no.=16
ω=2, It.no.=20


Blurred image
ω = 2.3, It.no.=4
ω = 2.3, It.no.=8
ω = 2.3, It.no.=12
ω = 2.3, It.no.=16
ω = 2.3, It.no.=20

vanCittert/Landweber iteration is simple and fast!

MORE INFORMATION:

Google: ”Inverse Problems”   1 850 000 hits!

Bibsys: ”Inverse Problems”   155 items

Wikipedia:
http://en.wikipedia.org/wiki/Inverse_problem
(incomplete)

Per Chr. Hansen’s home page:
http://www.imm.dtu.dk/~pch/ (recommended!)

Albert Tarantola's home page:
http://www.ipgp.fr/~tarantola/ (amusing!)