TMA 4180 Optimeringsteori
A worked example for the KKT theorem
Spring 2010

Objective function:

$$f(x) = 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2 \tag{1}$$

Constraints:

$$c_1(x) = 5 - x_1^2 - x_2^2 \geq 0, \tag{2}$$
$$c_2(x) = 6 - 3x_1 - x_2 \geq 0 \tag{3}$$

Since the objective function is continuous and Ω is finite (why?), we certainly have minima.

The Lagrange function:

$$\mathcal{L}(x, \lambda) = f(x) - \lambda_1 c_1(x) - \lambda_2 c_2(x). \tag{4}$$

The KKT-points are solutions of

$$\frac{\partial \mathcal{L}}{\partial x_1}(x, \lambda) = 4x_1 + 2x_2 - 10 + 2\lambda_1 x_1 + 3\lambda_2 = 0, \tag{5}$$
$$\frac{\partial \mathcal{L}}{\partial x_2}(x, \lambda) = 2x_1 + 2x_2 - 10 + 2\lambda_1 x_2 + \lambda_2 = 0, \tag{6}$$
$$\lambda_1\left(5 - x_1^2 - x_2^2\right) = 0, \tag{7}$$
$$\lambda_2(6 - 3x_1 - x_2) = 0, \tag{8}$$
$$\lambda_1, \lambda_2 \geq 0 \tag{9}$$

There are 4 possibilities for active constraints at the solution:

1. **No active constraints**
2. c_1 **active and** c_2 **inactive**
3. c_2 **active and** c_1 **inactive**
4. **Both** c_1 **and** c_2 **active**

Case 1: No active constraints

Must have $\lambda_1 = \lambda_2 = 0$, and the minimum will occur for a point where

$$\nabla \mathcal{L}(x, 0) = \nabla f(x) = 0. \tag{10}$$

Leads to:

$$4x_1 + 2x_2 - 10 = 0, \tag{11}$$
$$2x_1 + 2x_2 - 10 = 0, \tag{12}$$

Solution:

$$x_1^* = 0, \tag{13}$$
$$x_2^* = 5 \tag{14}$$

However,

$$c_1(x^*) = 5 - 0 - 5^2 = -20 \text{ (Violation!)} \tag{15}$$
$$c_2(x^*) = 6 - 0 - 5 = 1 \text{ (OK!)} \tag{16}$$

Case 4: Both constraints active

$$c_1(x) = 5 - x_1^2 - x_2^2 = 0, \tag{17}$$
$$c_2(x) = 6 - 3x_1 - x_2 = 0 \tag{18}$$

Quadratic equation for x_1:

$$10x_1^2 - 36x_1 + 31 = 0, \tag{19}$$

with two solutions (and two possible points:

$$x_a = (2.17... ,\ -0.52...)\,, \tag{20}$$
$$x_b = (1.42... ,\ 1.72...)\,. \tag{21}$$

We need to check the Lagrange multipliers ($\nabla_x \mathcal{L} = 0$):

$$4x_1 + 2x_2 - 10 + 2\lambda_1 x_1 + 3\lambda_2 = 0, \tag{22}$$
$$2x_1 + 2x_2 - 10 + 2\lambda_1 x_2 + \lambda_2 = 0. \tag{23}$$

Hence,

$$\lambda_1 = \frac{10 - 2x_2 - x_1}{3x_2 - x_1}, \lambda_2 = -\left(2x_1 + 2x_2 - 10 + 2\lambda_1 x_2\right) \tag{24}$$

The point x_a gives

$$\lambda_1 = -2.37...,\ \lambda_2 = 4.22... \tag{25}$$

Thus, x_a is unacceptable.

Similarly, the point x_b gives

$$\lambda_1 = 1.7...,\ \lambda_2 = -2.04... \tag{26}$$

Also x_b in unacceptable.

Case 3: c_1 inactive, c_2 active

Since c_2 is active:

$$6 - 3x_1 - x_2 = 0.$$

Thus,

$$x_2 = 6 - 3x_1, \tag{27}$$

and

$$f\left(x_1, 6 - 3x_1\right) = 5x_1^2 - 4x_1 - 24. \tag{28}$$

The (global) minimum occurs for $df/dx = 0$, or

$$x_1 = \frac{2}{5},\ x_2 = \frac{24}{5} \tag{29}$$

However,

$$c_1\left(x_1, x_2\right) = 5 - \left(\frac{2}{5}\right)^2 - \left(\frac{24}{5}\right)^2 = -\frac{91}{5} < 0! \tag{30}$$

We assumed that c_1 was inactive, but this is not a guarantee for not violating it!

Case 2: Only c_1 is active

$\lambda_2 = 0$:

$$\left(\frac{\partial \mathcal{L}}{\partial x_1} = \right) 4x_1 + 2x_2 - 10 + 2\lambda_1 x_1 = 0, \tag{31}$$
$$\left(\frac{\partial \mathcal{L}}{\partial x_2} = \right) 2x_1 + 2x_2 - 10 + 2\lambda_1 x_2 = 0, \tag{32}$$
$$x_1^2 + x_2^2 = 5. \tag{33}$$

Solution:

$$\begin{aligned} x_1^* &= 1, \\ x_2^* &= 2, \\ \lambda_1^* &= 1. \end{aligned} \tag{34}$$

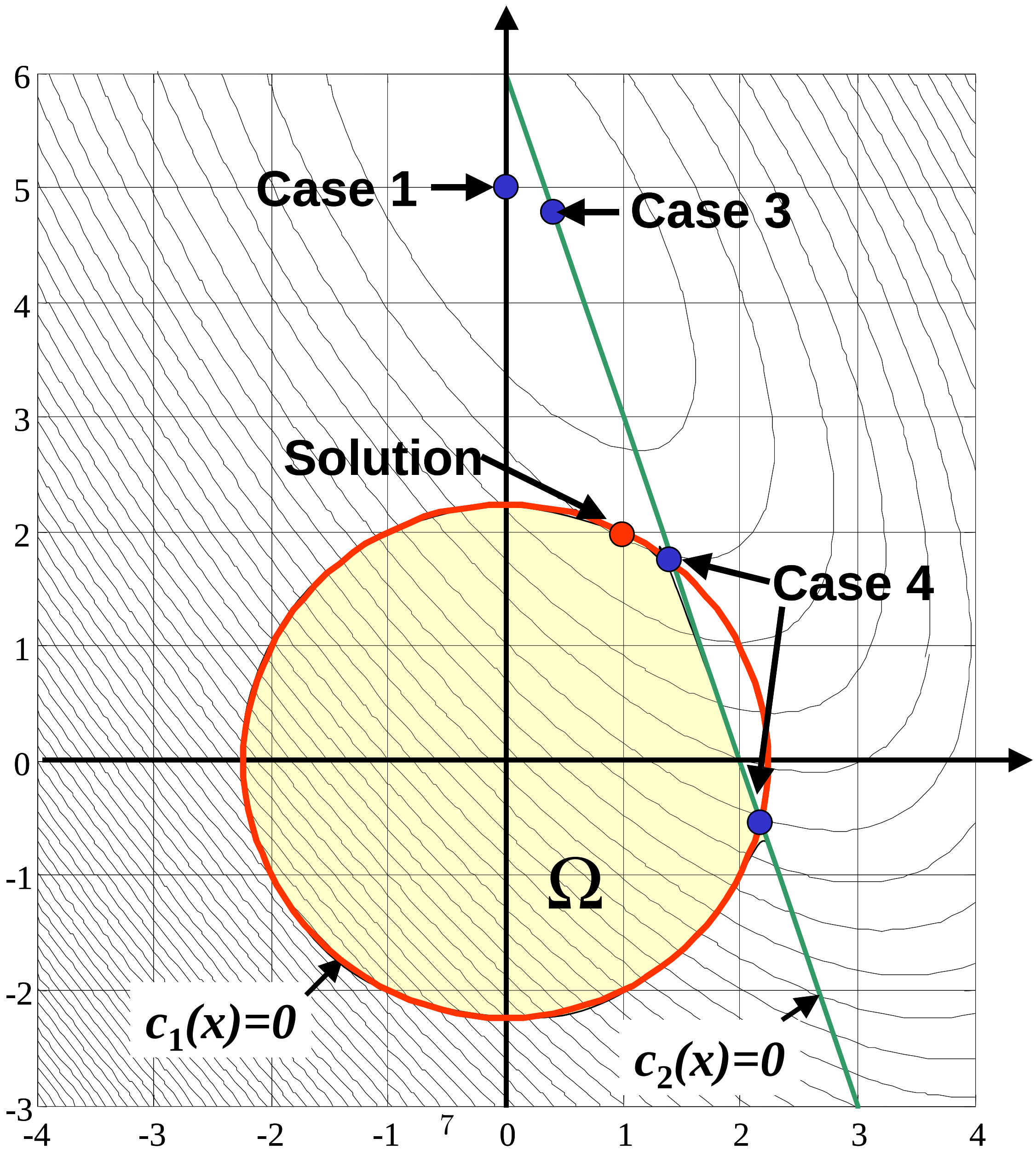
This looks promising, but we must also check c_2:

$$c_2\,(1,2) = 6 - 3 - 2 = 5 > 0 \quad \text{(OK!)}$$

(NB! There is also another solution of Eqns. 31 – 33. Find it, and prove it is NOT a KKT point!)

The only KKT-point is $(1,2)$, and since we know that a minimum exists, this is it!

Exercise: Consider convexity for this problem. Do we need to check the other point in Case 2?


Case 1
Case 3
Solution
Case 4
Ω
$c_1(x)=0$
$c_2(x)=0$
6
5
4
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3
4