THE LP PROBLEM IN STANDARD FORM

$$\min_{x \in \mathbb{R}^n} c'x,$$
$$Ax = b, \; x \geq 0.$$

- $x \geq 0$ means $x_i \geq 0, \; i = 1, \cdots, n$.
- A of size $r \times n$ is supposed to have *full rank* r.
- Ω is a **polytope (polyhedron** if bounded**).**
- This is a *convex* optimization problem $\Rightarrow$ KKT conditions sufficient for a global minimum.

GEOMETRY OF THE FEASIBLE SET

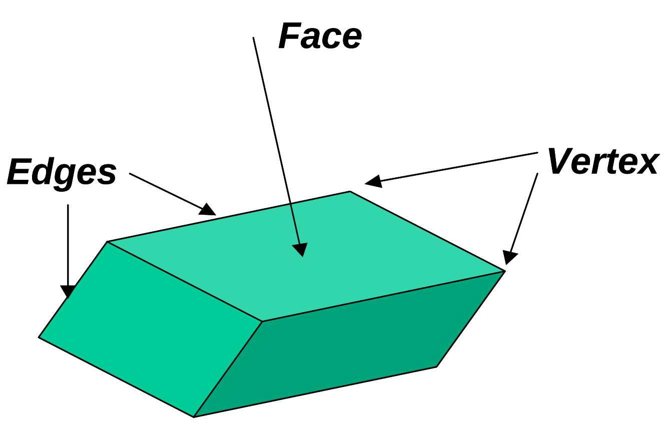
Definition: The point $x_e \ \in \partial\ \Omega$ ($=$ the boundary of Ω) is an *extreme point* if

$$x_e = \theta y + (1-\theta)\, z \ , \ \ y, z \in \Omega \ , \ \ 0 < \theta < 1$$

implies that $y = z = x_e$.

Where are the extreme points for a *line segment*, for $\mathbb{R}$ and $\mathbb{R}^n_+$, a *cube*, and a *sphere* (all sets closed)?

The extreme points for Ω are the *vertices*.



Definition: A feasible point x ($x \geq 0$, $Ax = b$) is called a *basic point* if there is an index set $\mathcal{B} = \{i_1, \cdots, i_r\}$, where the corresponding subset of columns of A,

$$\left\{a_{i_1}, \cdots, a_{i_r}\right\},$$

are linearly independent, and $x_i = 0$ for all $i \notin \mathcal{B}$.

If x_i happens to be 0 also for some $i \in \mathcal{B}$, we say that the basic point is *degenerate.*

For a basic point, the corresponding $r \times r$ matrix

$$B = \left[a_{i_1}, \cdots, a_{i_r}\right],$$

will be *non-singular*, and the equation $Bx_B = b$ has a unique solution.

The Fundamental Theorem for LP (N&W Theorem 13.2):

1. *If* $\Omega \neq \varnothing$, *it contains basic points.*

2. *If there are optimal solutions, there are optimal basic points (basic solutions).*

Theorem (N&W Theorem 13.3): *The basic points are the extreme points of* Ω.

The number of basic points is between 1 (because of the first statement in the Fundamental Theorem) and $\binom{n}{r}$.

THE SIMPLEX ALGORITHM

- The *Simplex Algorithm* is reported to have been discovered by G. B. Dantzig in 1947.

- The idea of the Simplex Algorithm is to search for the minimum by going from vertex to vertex (from basic point to basic point) in Ω.

- Hand calculations are *never used* anymore!

The Simplex Iteration Step

We assume that the problem has the standard form, and that we are located in a basic point which, after a rearrangement of variables, has the form

$$x = \left[\begin{array}{c} x_B \\ 0 \end{array} \right].$$

The partition is therefore according to $A = [B\ N]$, where B is non-singular, and

$$Ax = [B\ N] \begin{bmatrix} x_B \\ 0 \end{bmatrix} = Bx_B = b.$$

Split a general $x \in \Omega$ in the same way,

$$Ax = [B\ N] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Bx_1 + Nx_2 = b.$$

Hence,

$$x_1 = B^{-1}(b - Nx_2) = x_B - B^{-1}Nx_2.$$

Note also that

$$\begin{aligned} f(x) &= c'x = [c_1\ c_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= c_1'x_1 + c_2'x_2 \\ &= c_1'\left(x_B - B^{-1}Nx_2\right) + c_2'x_2 \\ &= c_1'x_B + \left(c_2' - c_1'B^{-1}N\right)x_2 \end{aligned}$$

Around $[x_B\ 0]'$, we may express both x_1 and $f(x)$ in terms of x_2.

We are located at $x_1 = x_B$, $x_2 = 0$, and try to change one of the components $(x_2)_j$ of x_2 so that

$$f(x) = c_1' x_B + \left(c_2' - c_1' B^{-1} N\right) x_2$$

decreases.

- If $\left(c_2' - c_1' B^{-1} N\right) \geq 0 \Rightarrow$ **FINISHED!**

Assume that $\left(c_2' - c_1' B^{-1} N\right)_j < 0$:

- If all components of x_1 increase when $(x_2)_j$ increases, then

$$\min c'x = -\infty.$$

$\Rightarrow$ **FINISHED!**

If not, we have the situation shown in Fig. 1.

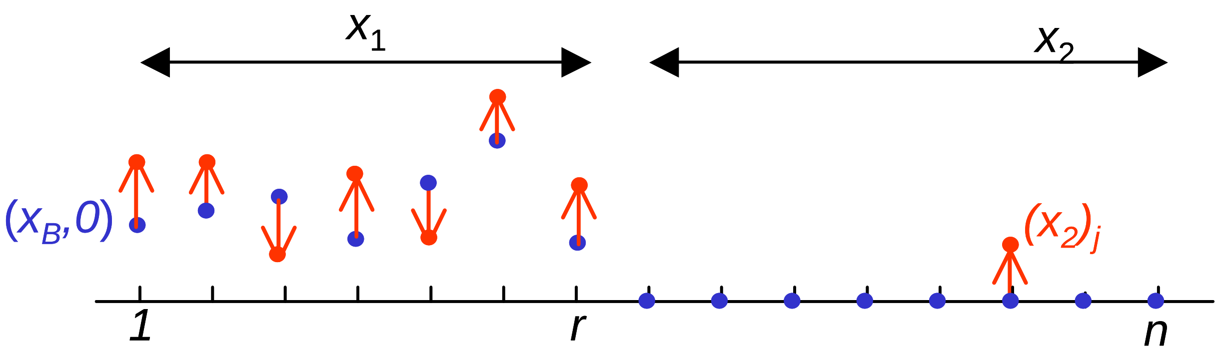


Figure 1: Change in x_1 when $(x_2)_j$ increases from 0.

- The Simplex algorithm always converges if all basic points are non-degenerate.

- Degenerate basic point: *Try a different component of* x_2. (**FINISHED** if impossible!)

- It is straightforward to construct a generalized Simplex Algorithm for bounds of the form

$$l_i \leq x_i \leq u_i, \; i = 1, \cdots, n.$$

- If we LU-factorize B once, we can update the factorization with the new column without making a complete new factorization (N&W, Sec. 13.4).

- It is often preferable to take the "steepest ridge" (fastest decrease in the objective) out from where we are (N&W, Sec. 13.5).

Starting the Simplex Method

The Simplex method consists of two phases:

- Phase 1: *Find a first basic point*
- Phase 2: *Solve the original problem*

The Phase 1 algorithm:

1. Turn the signs in $Ax = b$ so that $b \geq 0$.
2. Introduce additional variables $y \in \mathbb{R}^r$ and solve the extended problem

$$\min\,(y_1 + \cdots + y_r)\,,$$
$$[A \;\; I]\begin{bmatrix} x \\ y \end{bmatrix} = b, \quad x, y \geq 0.$$

   (Note that $[0\ b]'$ already is a basic point for the extended problem!).

Assume that the solution of the extended problem is

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

- If $y_0 \neq 0$, then the original problem is infeasible ($\Omega = \varnothing$).

- If $y_0 = 0$, then x_0 is a basic point ($=$ possible start for the original problem).

- This is not the only Phase 1 algorithm.

1 EPILOGUE

- Open Problem: *Are there LP algorithms of polynomial complexity*?

- The Simplex Method has exponential complexity in the worst case (*Kree–Minty–Cheval counterexample*)

- Interior Point Methods (Khatchiyan, 1978): $\#Op \propto \mathcal{O}\left(n^4 L\right)$

- Karmankar (1984): $\#Op \propto \mathcal{O}\left(n^{3.5} L\right)$

- Current record (?): *Interior Barrier Primal–Dual methods*, $\#Op \propto \mathcal{O}\left(n^3 L\right)$. (We return to this method after discussing penalty and barrier methods)

- Solving large LP problems is BIG business!

- Entering data into the computer for large LP problems is a lot of work. Look up a description of the industry standard "*MPS Data Format*" on the *internet.*

LINEAR PROGRAMMING IN
MATLAB OPTIMIZATION TOOLBOX
(may be a little outdated!)

Basic function: **linprog**

Solves the general LP-problem

$$\min_x \; f'x,$$
$$Ax \le b$$
$$A_{eq.}x = b_{eq.}$$
$$lb \le x \le ub$$

where f, x, b, b_{eq}, lb, and ub are vectors and A, A_{eq} are matrices (may be entered as *sparse* matrices)

Syntax:

```
x    =    linprog( f, A, b, Aeq, beq)
x    =    linprog( f, A, b, Aeq, beq, lb, ub)
x    =    linprog( f, A, b, Aeq, beq, lb, ub, x0)
x    =    linprog( f, A, b, Aeq, beq, lb, ub, x0, options)

[x,fval]                               = linprog(...)
[x,fval,exitflag]                      = linprog(...)
[x,fval,exitflag,output]               = linprog(...)
[x,fval,exitflag,output,lambda]        = linprog(...)
```

Example: The Standard form:

$$\min c' x,$$
$$Ax = b,$$
$$x \geq 0.$$

x = linprog(c,[ ],[ ],A,b,zeros(size(c)),[ ])

- Note the Matlab convention with *placeholders*, ”[ ]”

INPUT:

X0: Starting point. Used only for medium problems (*Nelder-Mead amoeba*).

Options: Structure of parameters

LargeScale:	'on'/’off’
Display:	'off'/'iter'/'final' (large scale problems)
MaxIter:	Max number of iterations
Simplex:	'on'/’off’ (‘on’ ignores x0)
TolFun:	Objective tolerance (large scale problems)

OUTPUT:

x,fval: Solution and objective

exitflag:

1	Iteration terminated OK
0	Number of iterations exceeded MaxIter
-2	No feasible point found
-3	Problem is unbounded
-4	NaN value encountered
-5	Both primal and dual are infeasible
-7	Search direction became too small

output: Structure of iteration information

iterations:	Number of iterations
algorithm:	Algorithm used
cgiterations:	The number of PCG iterations (large-scale algorithm only)
message:	Output message

lambda: Structure of Lagrange multipliers

ineqlin:	for linear inequalities $Ax \leq b$,
eqlin	for linear equalities $A_{eq}x = b_{eq}$,
lower	for lb,
upper	for ub.

ALGORITHMS:

Small/Medium scale: SIMPLEX-like including Phase 1

Large scale: Primal-dual inner method

EXAMPLES FROM THE DOCUMENTATION

A. Small Problem

Find ***x*** that minimizes

$$f(x) = -5x_1 - 4x_2 - 6x_3$$

subject to

$$x_1 - x_2 + x_3 \le 20$$
$$3x_1 + 2x_2 + 4x_3 \le 42$$
$$3x_1 + 2x_2 \le 30$$
$$0 \le x_1, 0 \le x_2, 0 \le x_3$$

First, enter the coefficients, then call **LINPROG**:

```
f = [-5  -4  -6]';
A =  [ 1 -1  1
       3  2  4
       3  2  0 ];
b = [20  42 30]';
lb = zeros(3,1);
[x,fval,exitflag,output,lambda] = … linprog(f,A,b,[],[],lb);
```

```
    x               =     [0  15  3]
    fval            =     -78.0
    output:
            iterations:   6
            algorithm:    'large-scale: interior point' (!)
            cgiterations: 0
            message:      'Optimization terminated.'

    lambda.ineqlin     =      [0  1.5  0.5]
    lambda.lower       =      [1  0  0]
```

For solution by the Simplex method:

```
f = [-5  -4  -6]';
A =  [ 1 -1  1
       3  2  4
       3  2  0 ];
b = [20  42 30]';
lb = zeros(3,1);
options = optimset('LargeScale','off','Simplex','on');
[x,fval,exitflag,output,lambda] = ...
linprog(f,A,b,[],[],lb,[],[],options);
```

(NB! If you forget enough placeholders, [ ] , you get the error message ”LINPROG only accepts inputs of data type double”)

Now **output** gives:

```
  iterations:        3
  algorithm:         'medium scale: simplex'
  cgiterations:      []
  message:           'Optimization terminated.'
```

(same solution!)

B Medium Problem

This problem is stored as a Matlab MAT-file.

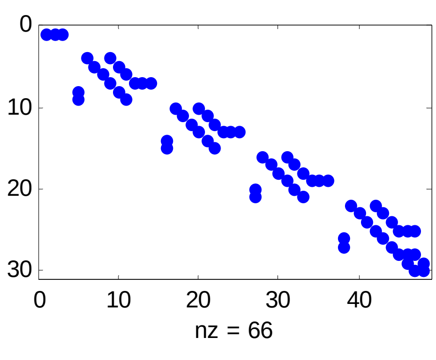
- 48 unknowns
- 30 inequality constraints
- 20 equality constraints
- $x \geq 0$

Entered into Matlab simply by

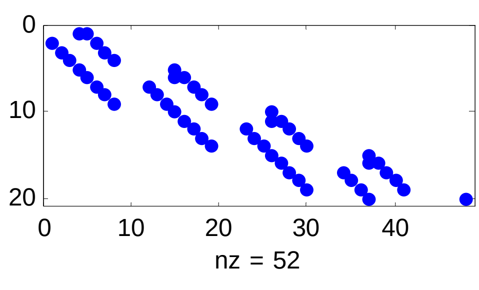
load sc50b

A	30x48	(sparse)
Aeq	20x48	(sparse)
b	30x1	
beq	20x1	
f	48x1	
lb	48x1	

Sparsity patterns:



A (inequaliti~~t~~es)



A_{eq} (equalities)

⇒load sc50b

```
options = optimset('LargeScale','off','Simplex','on');
[x,fval,exitflag,output,lambda] =  ...
    linprog(f,A,b,Aeq,beq,lb,[],[],options);
```

x = [30 28 42 ... 102.4870]

Only lambda.ineqlin(2) and lambda.ineqlin(3) equal to 0:

only inequality 2 and 3 non-active.

max(lambda.lower)= 8.2808e-015 $\Longrightarrow$ $x_i > 0$ for $i = 1, \cdots, 48$.

```
output =
    iterations:    43
    algorithm:     'medium scale: simplex'
    cgiterations:  []
    message:       'Optimization terminated.'
```

Large scale option:

```
options = optimset('LargeScale','on');
[x,fval,exitflag,output,lambda] = ...
                    linprog(f,A,b,Aeq,beq,lb,[],[],options);
```

```
output =
    iterations:    8
    algorithm:     'large-scale: interior point'
    cgiterations:  0
    message:       'Optimization terminated.'
```

Same solution!

With display of results for each iteration:

options = optimset('LargeScale','on','Display','iter');

```
  Residuals:   Primal     Dual     Duality    Total
               Infeas    Infeas      Gap       Rel
               A*x-b   A'*y+z-f     x'*z      Error
  --------------------------------------------------
  Iter    0:  1.50e+03  2.19e+01  1.91e+04  1.00e+02
  Iter    1:  1.15e+02  3.18e-15  3.62e+03  9.90e-01
  Iter    2:  8.32e-13  1.96e-15  4.32e+02  9.48e-01
  Iter    3:  3.51e-12  1.87e-15  7.78e+01  6.88e-01
  Iter    4:  1.81e-11  3.50e-16  2.38e+01  2.69e-01
  Iter    5:  2.63e-10  1.23e-15  5.05e+00  6.89e-02
  Iter    6:  5.88e-11  2.72e-16  1.64e-01  2.34e-03
  Iter    7:  2.61e-12  2.59e-16  1.09e-05  1.55e-07
  Iter    8:  7.97e-14  5.67e-13  1.09e-11  3.82e-12
Optimization terminated.
```

FOR MORE INFO: Read documentation of **linprog!**

OPTIMIZATION SOFTWARE – 2010

http://wiki.mcs.anl.gov/NEOS/index.php/NEOS_Wiki

(NEOS = Network-Enabled Optimization System)

- AIMMS modeling system
- AMPL modeling language.
- ANALYZE linear programming model analysis.
- APOPT - nonlinear programming.
- APMonitor modeling language.
- ASA - adaptive simulated annealing.
- BPMPD - linear programming.
- BQPD - quadratic programming.
- BT - minimization.
- BTN - block truncated Newton.
- CBC - mixed-integer linear programming.
- CML - constrained maximum likelihood.
- CNM - linear algebra and minimization.
- CO - constrained optimization.
- COMPACT - design optimization.
- CONOPT - nonlinear programming.
- CONSOL-OPTCAD - engineering system design.
- CONTIN - systems of nonlinear equations.
- CLP - linear programming.
- CPLEX - linear programming.
- C-WHIZ - linear programming models.
- DATAFORM - model management system.
- DFNLP - nonlinear data fitting.
- DOC - Design Optimization Control Program.
- DONLP2 - nonlinear constrained optimization.
- DOT - Design Optimization Tools.
- EASY FIT - parameter estimation in dynamic systems.
- Excel and Quattro Pro Solvers - spreadsheet-based linear, integer and nonlinear programming
- EZMOD - modeling environment for decision support systems
- FortMP - linear and mixed integer quadratic programming.
- FSQP - nonlinear and minmax constrained optimization, with feasible iterates.
- GAMS - General Algebraic Modeling System.
- GAUSS - matrix programming language.
- GENESIS - structural optimization software.
- GENOS 1.0 - nonlinear network optimization.
- GINO - nonlinear programming.
- GRG2 - nonlinear programming.
- GOM - Global Optimization for Mathematica.
- GUROBI - linear programming.
- HOMPACK - nonlinear equations and polynomials.
- HOPDM - linear programming (interior-point).
- HARWELL Library - linear and nonlinear programming, nonlinear equations, data fitting.
- HS/LP Linear Optimizer - linear programming.
- ILOG - constraint-based programming and nonlinear optimization.
- IMSL - Fortran and C Library.
- IPOPT - nonlinear programming.
- KNITRO - nonlinear programming.
- KORBX - linear programming.
- LAMPS - linear and mixed-integer programming.
- LANCELOT - large-scale problems.
- LBFGS - unconstrained minimization.
- LBFGS-B - bound-constrained minimization.
- LGO IDE - continuous and Lipschitz global optimization.

- LINDO - linear, mixed-integer and quadratic programming.
- LINGO - modeling language.
- LIPSOL - linear programming.
- LNOS - linear programming/network flow problems.
- LOQO - Linear programming, unconstrained and constrained nonlinear optimization.
- LP88 and BLP88 - linear programming.
- LSGRG2 - nonlinear programming.
- LSNNO - large scale optimization.
- LSSOL - least squares problems.
- M1QN3 - unconstrained optimization.
- MATLAB - optimization toolbox.
- MAXLIK - maximum likelihood estimation.
- MCS - global optimization.
- MILP88 - mixed integer programming.
- MINOS - linear programming and nonlinear optimization.
- MINTO - mixed integer linear programming.
- MINPACK-1 - nonlinear equations and least squares.
- MIPIII - mixed integer programming.
- MODFIT - parameter estimation in dynamic systems.
- MODLER - linear programming modeling language.
- MODULOPT - unconstrained problems and simple bounds.
- MOSEK - linear programming and convex optimization.
- MPL - modeling system
- MPSIII - mathematical programming system.
- NAG C Library - nonlinear and quadratic programming, minimization
- NAG Fortran Library - nonlinear and quadratic programming, minimization
- NETFLOW - network optimization.
- NITSOL - systems of nonlinear equations.
- NLopt - local and global nonlinear optimization, including nonlinear constraints, with and without user-supplied gradients
- NLPE - minimization and least squares problems.
- NLPJOB - Mulicriteria optimization.
- NLPQL - nonlinear programming.
- NLPQLB - nonlinear programming with constraints.
- NLSSOL - constrained nonlinear least squares problems.
- NLPSPR - nonlinear programming.
- NOVA - nonlinear programming.
- NPSOL - nonlinear programming.
- ODRPACK - NLS and ODR problems.
- OML - linear and mixed-integer programming, model management.
- OPL Studio - optimization language and solver environment.
- OPTDES - design optimization tool.
- OPTECH - global optimization.
- OptiA - unconstrained, constrained, quadratic, minimax, nonsmooth, and global optimization
- OPTIMA Library - optimization and sensitivity analysis.
- OPTIMAX - component software for optimization
- OPTMUM - optimization.
- OPTPACK - constrained and unconstrained optimization.
- OptQuest - global optimization
- OSL - linear, quadratic and mixed-integer programming.
- PCOMP - modelling language with automatic differentiation.
- PCx - linear programming with a primal-dual interior-point method.
- PDEFIT - parameter estimation in partial differential equations.
- PETSc - parallel solution of nonlinear equations and unconstrained minimization problems.
- PLAM - algebraic modeling language for mixed integer programming, constraint logic programming, etc.
- PORT 3 - minimization, least squares, etc.
- PROC LP - linear and integer programming.
- PROC NETFLOW - network optimization.

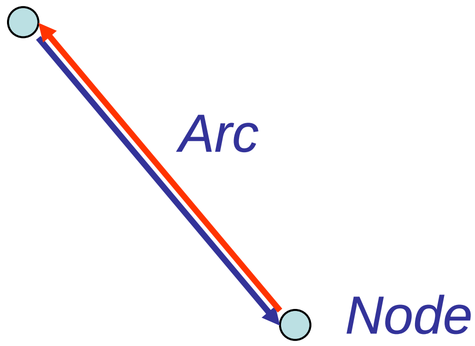
- PROC NLP - various quadratic and nonlinear optimization problems.
- PROPT - optimal control software for MATLAB users.
- Q01SUBS - quadratic programming for matrices.
- QAPP - quadratic assignment problems.
- QL - quadratic programming.
- QPOPT - linear and quadratic problems.
- RANDMOD - linear programming model randomizer.
- SCIP - mixed-integer linear programming.
- SIMUSOLV - modeling software.
- SPRNLP - sparse and dense nonlinear programming, sparse nonlinear least squares, including the SOCS package for optimal control
- SPEAKEASY - numerical problems and operations research.
- SNOPT - large-scale quadratic and nonlinear programming problems.
- SQOPT - large-scale linear and convex quadratic programming problems.
- SQP - nonlinear programming.
- SYMPHONY - mixed-integer linear programming.
- SYNAPS Pointer - multidisciplinary design optimization software
- SYSFIT - parameter estimation in systems of nonlinear equations.
- TENMIN - unconstrained optimization.
- TENSOLVE - nonlinear equations and least squares.
- TN/TNBC - minimization.
- TNPACK - nonlinear unconstrained minimization.
- TSA88 - network linear programming.
- TOMLAB - Matlab Optimization.
- UNCMIN - unconstrained optimization.
- VE08 - nonlinear optimization.
- VE10 - nonlinear least squares.
- VIG and VIMDA - decision support system.
- What'sBest - linear and mixed integer programming.
- WHIZARD - linear programming, mixed-integer programming.
- XLSOL - Linear, integer and nonlinear programming for AMPL models
- XPRESS-MP from Dash Associates - linear and integer programming.

TMA 4180 Optimeringsteori

Minimum Cost Network Flow Analysis Using LP

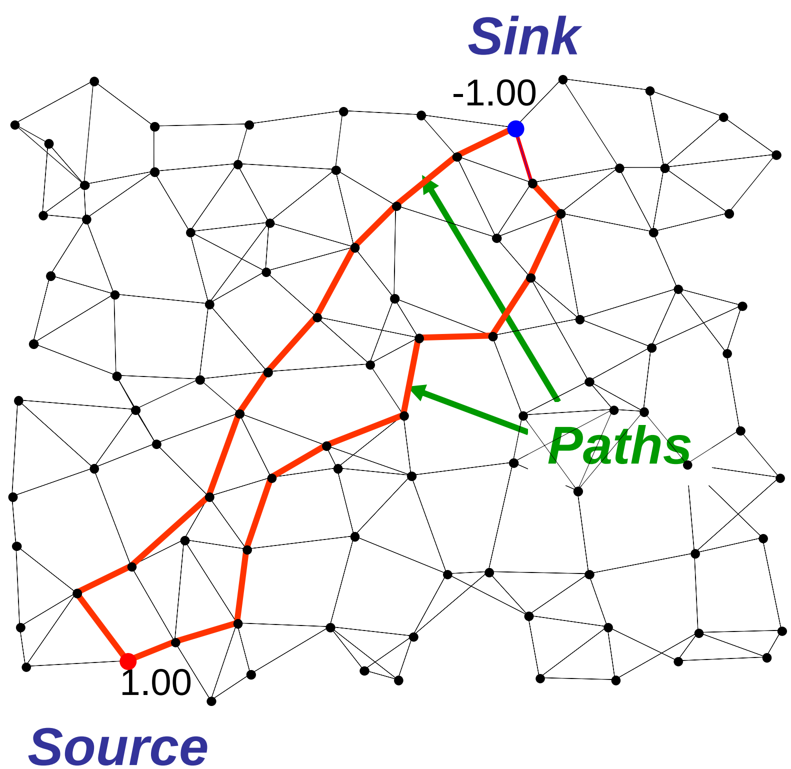
Harald E. Krogstad

March 2007



An arc is characterized by

- Prize pr. flow unit along arc
- Lower bound (for initiating transport)
- Upper bound (capacity)



Sources: (Production/providers)
- Capacity
- *Cost* pr. unit delivered to the network

Sinks (Consumers/receivers):
- Capacity
- *Income* to network from deliveries

Source: Production *b>0.*
Sink: Absorption, *b < 0.*

Variables $x=\{x_i\}, x_i \geq 0.$ (flow in the arcs)

NB! 2 variables for each arc: 2 *directions*

Node: $$\sum_{\text{inflow}} x_i = \sum_{\text{outflow}} x_i$$

Source/Sink: $$b_s = \sum_{\text{outflow}} x_i - \sum_{\text{inflow}} x_i$$

A *balanced* network: $$\sum_{\text{Sources/sinks}} b_s = 0$$

Price for delivery: $$f(x) = \sum_{\text{arcs}} c_i x_i = c' x$$

Cost for one unit along arc "*i*": $\{c_i\}$

Upper bound on capacity for arc "*i*": $\{ub_i\}$

Lower bound on capacity for arc "*i*": $\{lb_i\}$

The LP formulation:

$$\min_x c'x$$
$$\sum_{\text{outflow}} x_i - \sum_{\text{inflow}} x_i = b_n,\ n = 1, \ldots, Nodes,$$
$$lb \le x \le ub.$$

$$\min_x c'x$$
$$A_{eq}x = b_{eq}$$
$$lb \le x \le ub$$

The matrix is a *sparse* matrix with only -1, 0, and -1

Simsys_sparse

MATLAB® CENTRAL

An open exchange for the MATLAB and Simulink user community

http://www.mathworks.com/matlabcentral/

Per Bergström
Luleå University of Technology

RANDOM NETWORK GENERATION

Prescribe:

- Numbers of sources and sinks
- Max number of arcs around one node
- Min number of arcs around one node
- Random upper bound
- Distribution of nodes
- Interactive network modification
- Random costs

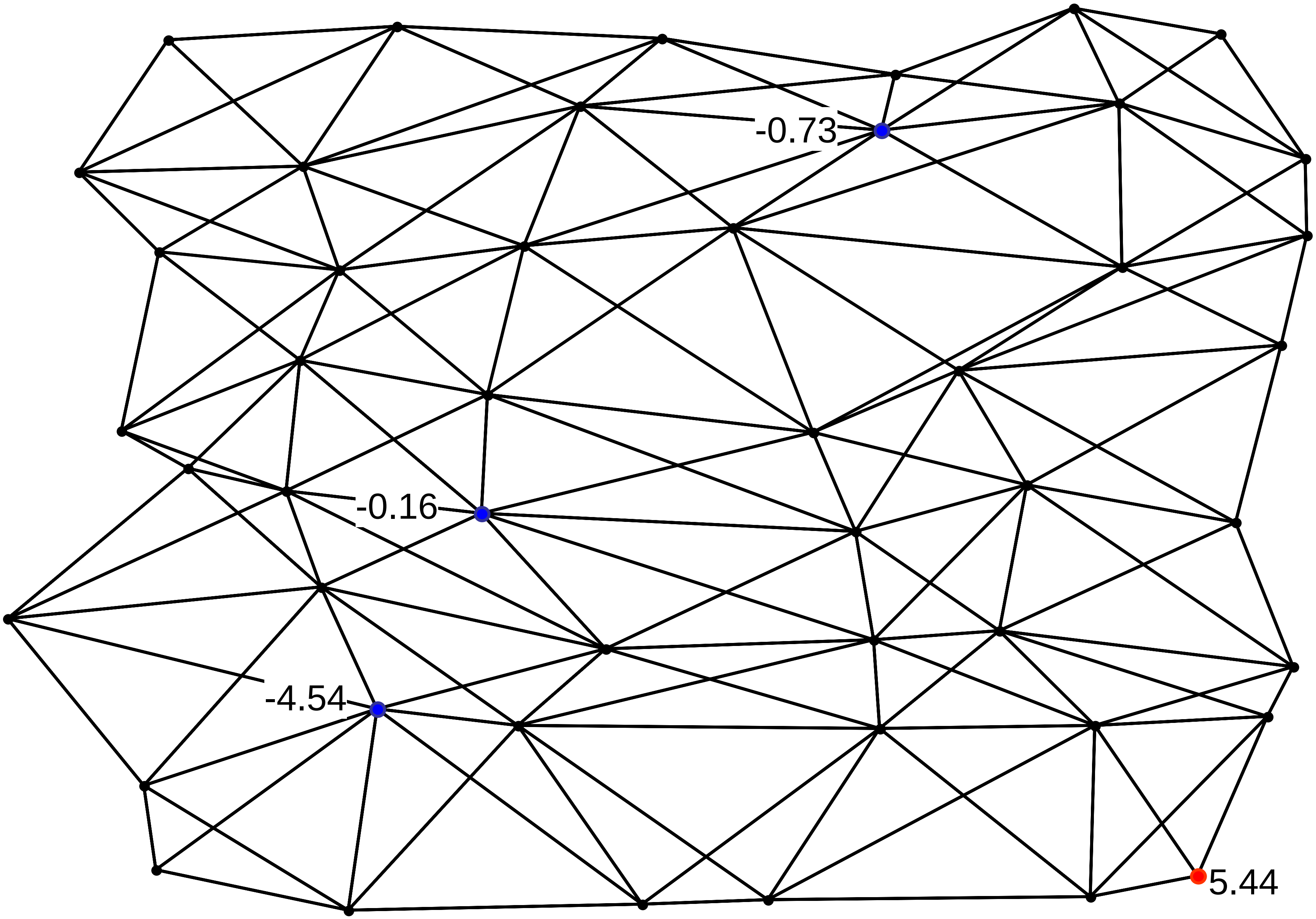
The algorithm provides:

- Number of nodes
- Upper bound of capacity
- A_{eq} matrix
- Balanced production/consumption at the sources and sinks

```
[Aeq,beq,lb,ub,c]=simsys_sparse(100);
```

Solution in Matlab: x = linprog(c,[],[],Aeq,beq,lb,ub)

RANDOMLY GENERATED NETWORK



The LP-problem:

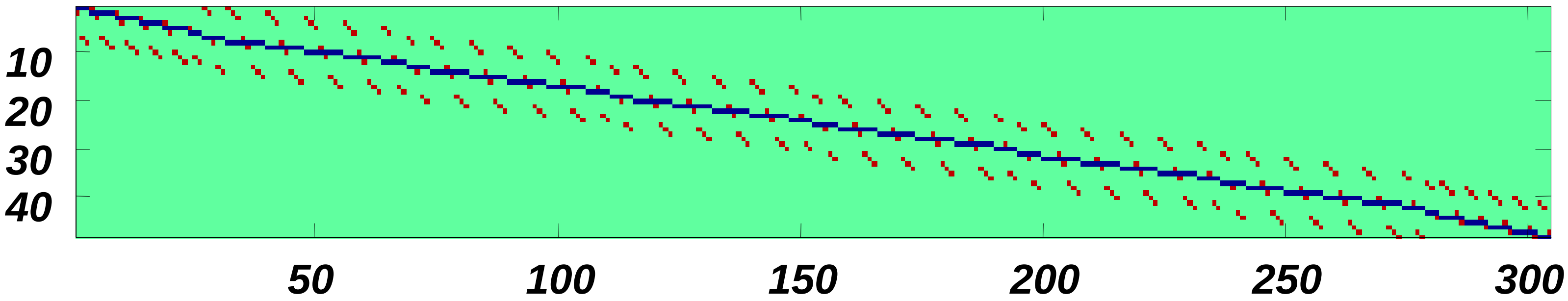
• Number of arcs: 304

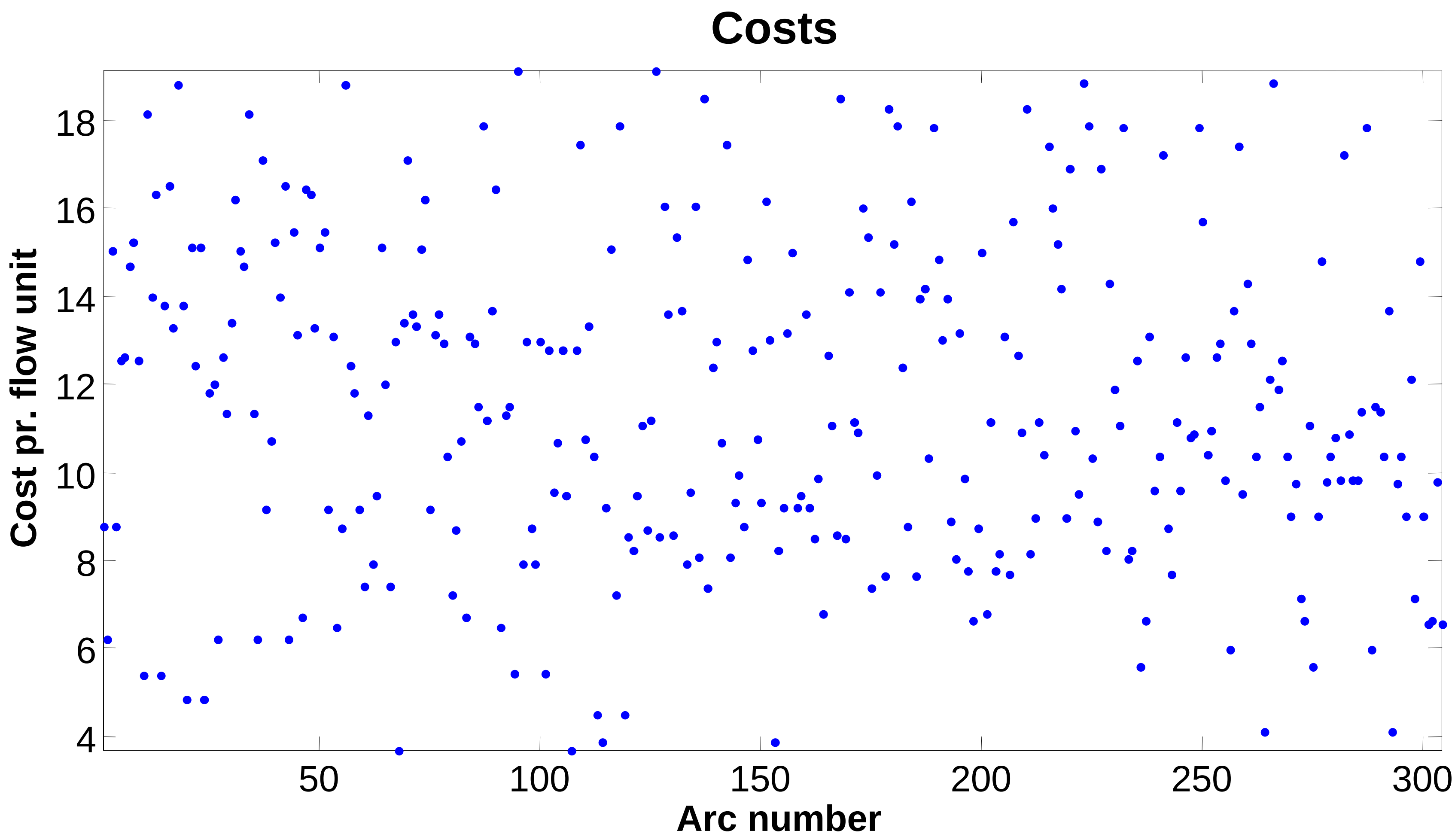
• Lower bounds: 0

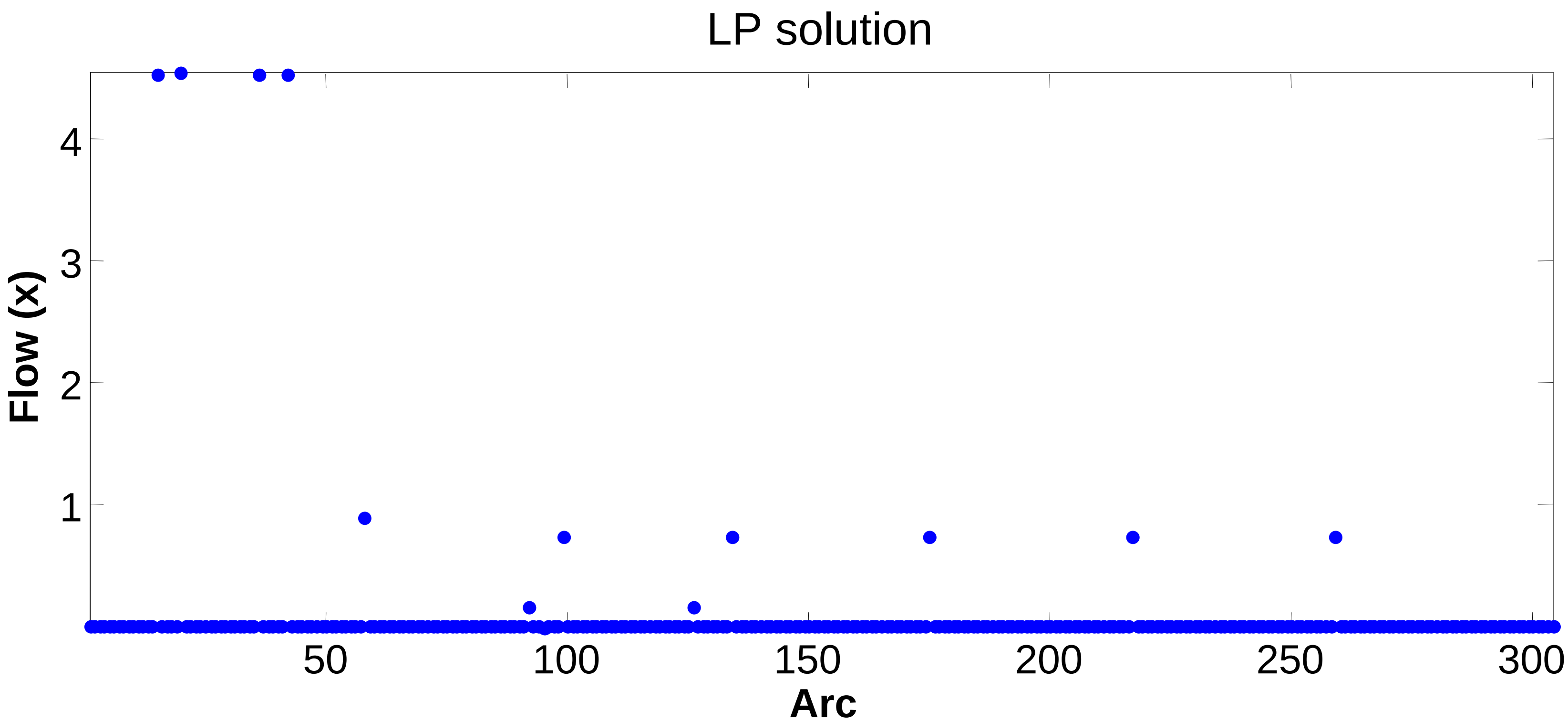
• Upper bounds: -

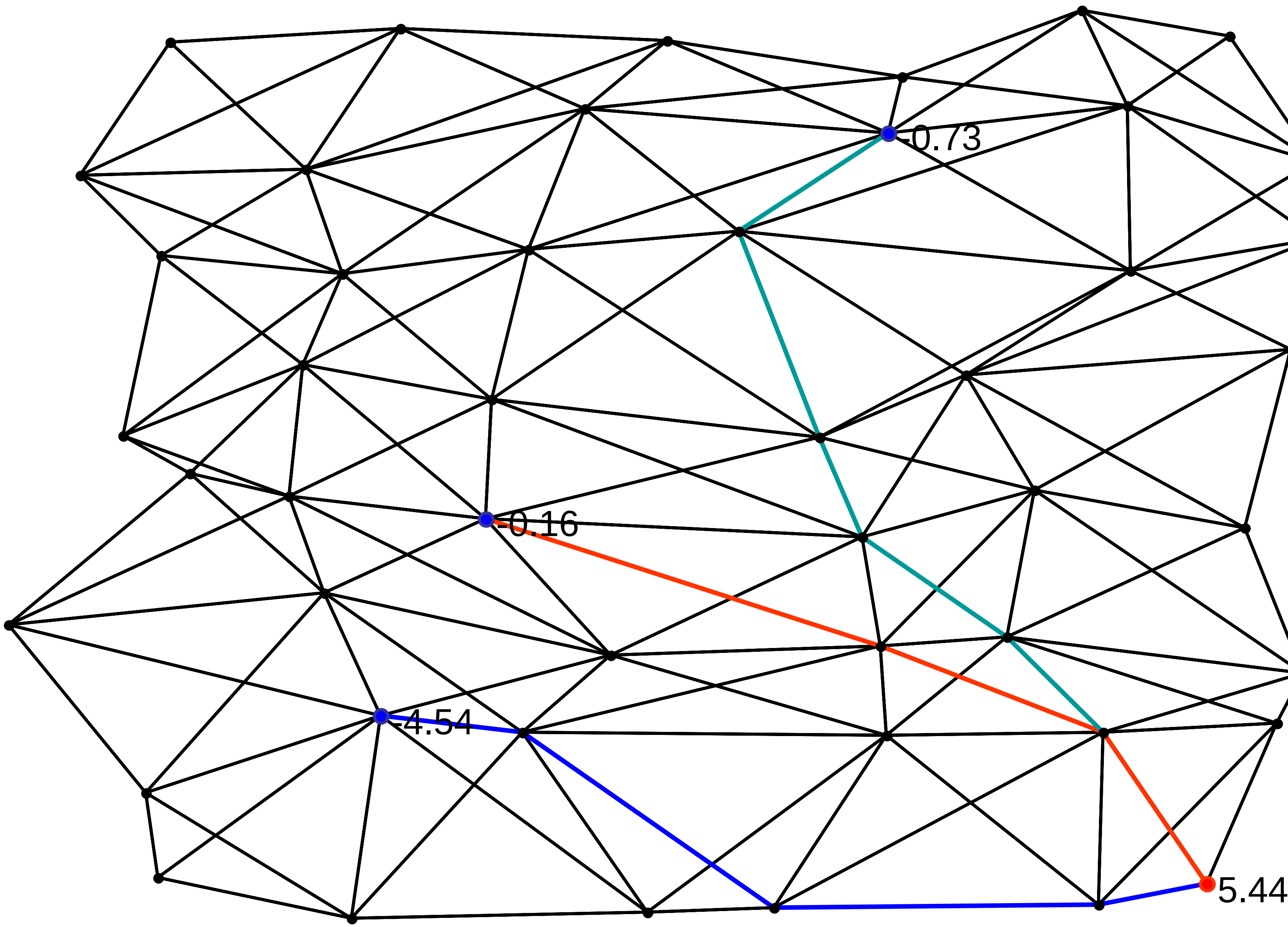
• Equality constraints: 48

A_{eq}-matrix:

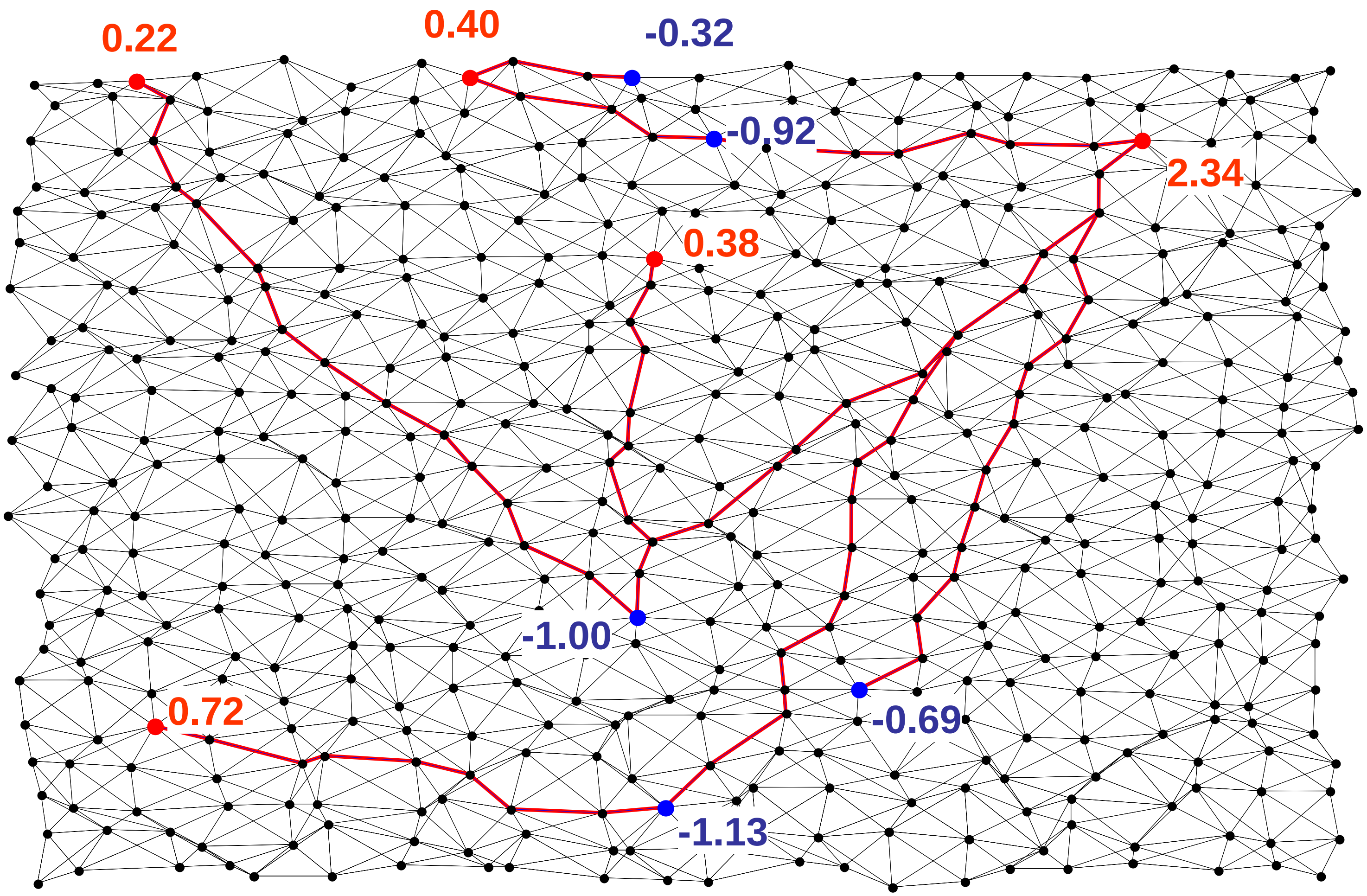



Costs
Cost pr. flow unit
Arc number
4
6
8
10
12
14
16
18
50
100
150
200
250
300


LP solution
Flow (x)
Arc
1
2
3
4
50
100
150
200
250
300

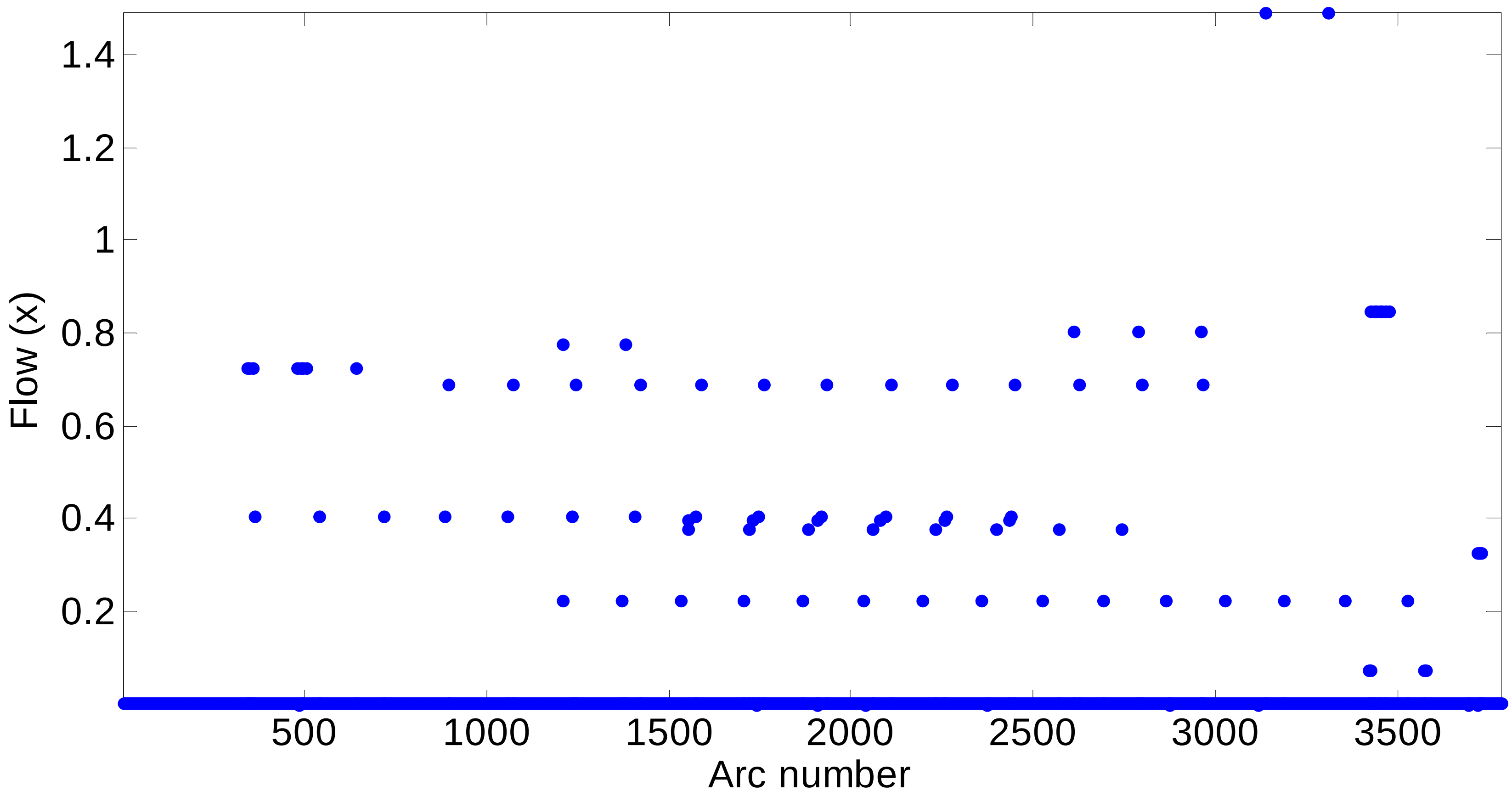


-0.73
-0.16
-4.54
5.44



$$\dim(x) = 3782$$

$$\dim(A_{eq}) = 506 \times 3782$$



1.4
1.2
1
0.8
0.6
0.4
0.2
Flow (x)
500
1000
1500
2000
2500
3000
3500
Arc number

Practical Optimization: A Gentle Introduction

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(*very* soft introduction ☺)