Moving the Derivative Inside the Integral

When computing functional derivatives we often need to interchange derivatives and integrals, and usually this works fine, but in some cases the result will not be what we expect. Engineers may write "if we may change the order of derivation and integration, then ...", but this is not acceptable in a mathematical text. A simple but important counterexample is discussed at the end of the note.

Let us consider the equation

$$\frac{d}{dt} \int_a^b f(x,t)\, dx = \int_a^b \frac{\partial f(x,t)}{\partial t}\, dx. \tag{1}$$

The left hand side is the derivative of the function

$$\Phi(t) = \int_a^b f(x,t)\, dx, \tag{2}$$

that is, $\frac{d\Phi}{dt}(t)$. On the right hand side, the function $\frac{\partial f(x,t)}{\partial t}$ has its usual meaning, and the right hand side is equal to another function of t, say $\Psi(t)$. Eq. 1 then states that

$$\Phi'(t) = \Psi(t), \tag{3}$$

but when is this really true?

Let us consider the derivative of $\Phi(t)$:

$$\begin{aligned}
\Phi'(t) &= \lim_{\tau \to 0} \frac{\Phi(t+\tau) - \Phi(t)}{\tau} \\
&= \lim_{\tau \to 0} \frac{\int_a^b f(x,t+\tau)\, dx - \int_a^b f(x,t)\, dx}{\tau} \\
&= \lim_{\tau \to 0} \int_a^b \frac{f(x,t+\tau) - f(x,t)}{\tau}\, dx.
\end{aligned} \tag{4}$$

Here it is tempting to move $\lim_{\tau \to 0}$ inside the integral since

$$\frac{\partial f(x,t)}{\partial t} = \lim_{\tau \to 0} \left(\frac{f(x,t+\tau) - f(x,t)}{\tau} \right). \tag{5}$$

Unfortunately, this is not always valid, even if the limit in Eq. 5 should exist for all x. However, Lebesgue integration theory has a very powerful criterion called *Lebesgue Dominated Convergence Theorem (LDCT)*. This tells us that if the limit in Eq. 5 exists for *almost all* x, and there is a function $H(x) \geq 0$, $\int_a^b H(x)\, dx < \infty$, such that

$$\left| \frac{f(x,t+\tau) - f(x,t)}{\tau} \right| \leq H(x), \tag{6}$$

then Eq. 1 holds.

This criterion is very general, and more than we actually need most of the time. Assume therefore that we deal with finite intervals and that f and $\partial f/\partial t$ are continuous for $(x,t) \in [a,b] \times [\alpha,\beta]$, say. Finding a suitable H is then trivial. Applying the *Secant Formula*, and the finite maximum of continuous functions on a bounded set, we obtain

$$\left| \frac{f(x,t+\tau) - f(x,t)}{\tau} \right| = \left| \frac{\partial f(x,t+\theta\tau)}{\partial t} \right| \leq \max_{x} \left| \frac{\partial f(x,t+\theta\tau)}{\partial t} \right|$$

$$\leq \max_{x,t} \left| \frac{\partial f(x,t)}{\partial t} \right| = M < \infty, \ 0 < \theta < 1. \tag{7}$$

Thus, $H(x) = M$ will do, since $\int_a^b M\,dx < \infty$. This argument actually proves a convenient theorem stated in Troutman, p. 426:

Theorem A13: *If f and $\partial f/\partial t$ are continuous for $(x,t) \in [a,b] \times [\alpha,\beta]$, and $[a,b]$ is finite, then the function*

$$\Phi(t) = \int_a^b f(x,t)\,dx \tag{8}$$

is differentiable and

$$\frac{d\Phi(t)}{dt} = \frac{d}{dt} \int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f(x,t)}{\partial t}\,dx. \tag{9}$$

(In fact, $\frac{d\Phi(t)}{dt}$ is even continuous by another application of LDCT).

A somewhat subtle counter-example, well-known in the theory of shock-waves, is the function

$$h(x,t) = \begin{cases} 1 & x < t, \\ 0 & x > t. \end{cases} \tag{10}$$

For $t \in (0,1)$,

$$\Phi(t) = \int_0^1 h(x,t)\,dx = \int_0^t 1 \cdot dx = t, \tag{11}$$

and therefore

$$\frac{d\Phi(t)}{dt} = \frac{d}{dt} \int_0^1 h(x,t)\,dx = 1. \tag{12}$$

On the other hand,

$$\frac{\partial h(x,t)}{\partial t} = 0 \tag{13}$$

for all $t \neq x$, and hence,

$$\Psi(t) = \int_0^1 \frac{\partial h(x,t)}{\partial t}\,dx = \int_0^t \frac{\partial h(x,t)}{\partial t}\,dx + \int_t^1 \frac{\partial h(x,t)}{\partial t}\,dx = 0. \tag{14}$$

Thus, we need to be careful when $h(x,t)$ is not continuous!

Let us finish with a more positive example:

$$J(y) = \int_0^1 \sin(y(x))\,dx, \ y \in C[0,1]. \tag{15}$$

Then
$$J(y + \varepsilon v) = \int_0^1 \sin(y(x) + \varepsilon v(x)) \, dx, \tag{16}$$

Since both $\sin(y(x) + \varepsilon v(x))$ and $\frac{\partial}{\partial \varepsilon} \sin(y(x) + \varepsilon v(x)) = \cos(y(x) + \varepsilon v(x)) v(x)$ are continuous on, say $[0,1] \times [-1,1]$, we apply Thm. A13 to conclude that

$$\frac{d}{d\varepsilon} \left(\int_0^1 \sin(y(x) + \varepsilon v(x)) \, dx \right) \Big|_{\varepsilon=0} \tag{17}$$

$$\overset{\text{Th}^m \text{A}13}{=} \int_0^1 \frac{\partial \sin(y(x) + \varepsilon v(x)) \, dx}{\partial \varepsilon} \Big|_{\varepsilon=0} dx \tag{18}$$

$$= \int_0^1 \cos(y(x)) v(x) \, dx. \tag{19}$$