TMA 4180 OPTIMERINGSTEORI

REVISION SPRING 2012

THE CONGUGATE GRADIENT METHOD

- TESTS AND CONVERGENCE

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CONVERGENCE TESTS

Generatating the A matrix

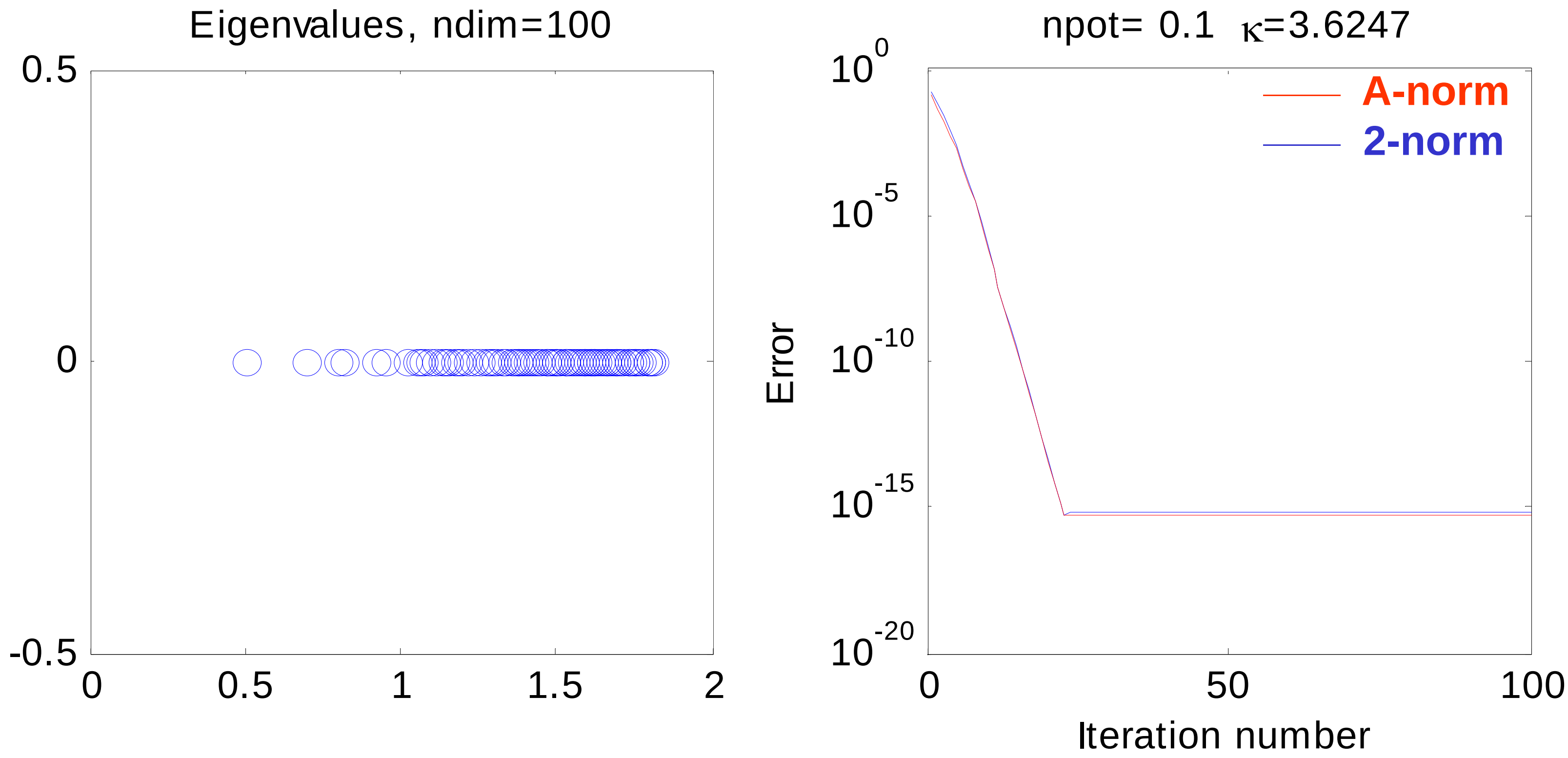
```
% Define matrix
ndim = 100;
R    = randn(ndim);  % Random matrix
npot = .5            % Steers the eigenvalue
A  = (R'*R)^npot;    % A matrix, >0.
lamb = eig(A);       % Eigenvalues
kappa= max(lamb)/min(lamb) % The condition number
xsol = rand(ndim,1); %The solution
b    = A*xsol;       % The b-vector

Norm2 = sqrt(xsol'*xsol);
NormA = sqrt(xsol'*A*xsol);
```

The C-G algorithm

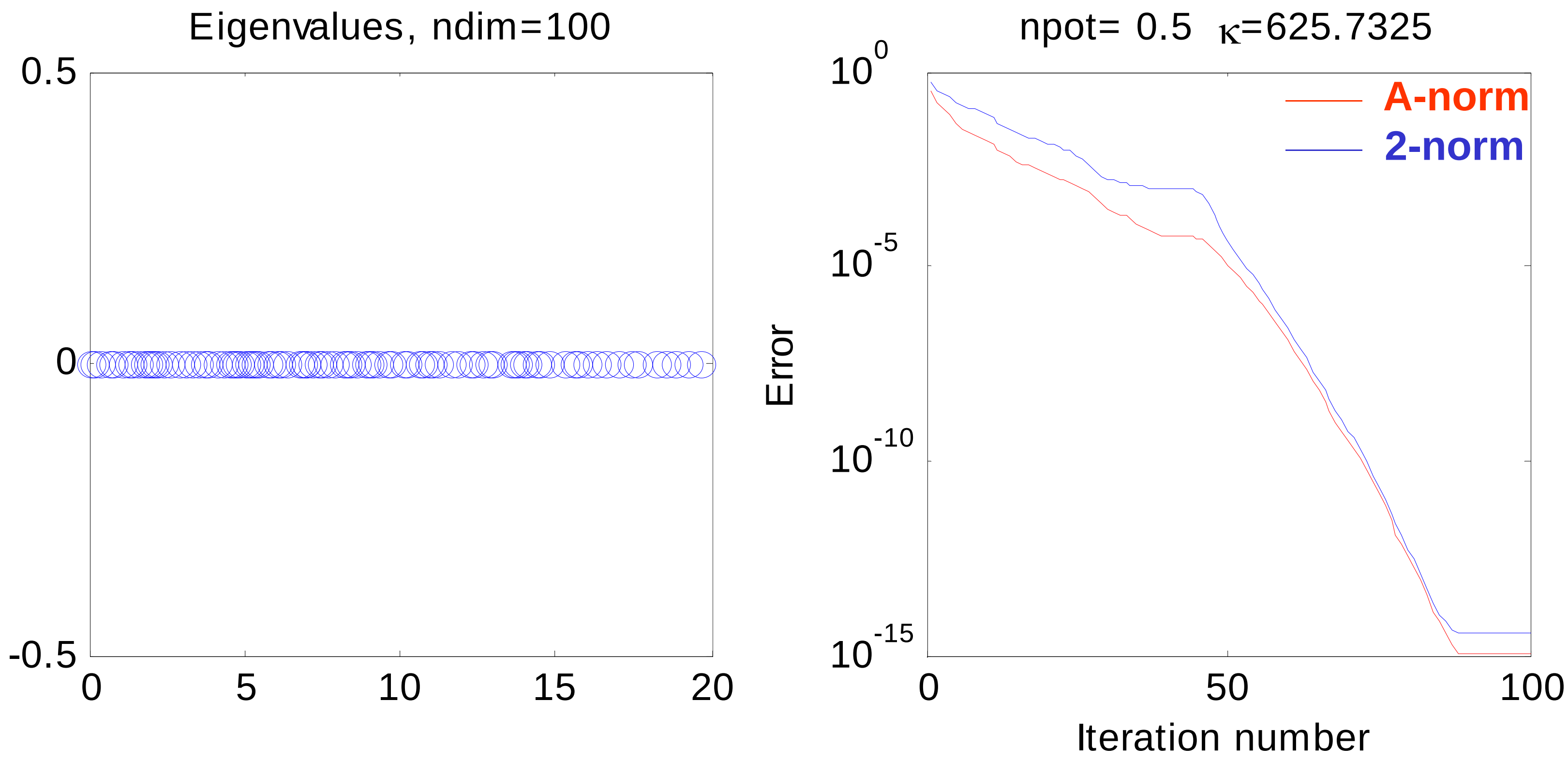
```
x = zeros(size(b));
g = -b; p = -g;
%
for loop = 1:ndim
   Ap   = A*p;
   alfa = -(p'*g)./(p'*Ap);
   x    = x + alfa*p;
   g    = g + alfa*Ap; %  g = A*x-b;
   beta = (g'*Ap)./ (p'*Ap);
   p    = -g + beta*p;
   err2(loop) = sqrt((x-xsol)'*(x-xsol))/Norm2;
   errA(loop) = sqrt((x-xsol)'*A*(x-xsol))/NormA;
end;
```

Note: Every iteration requires only one matrix/vector operation!

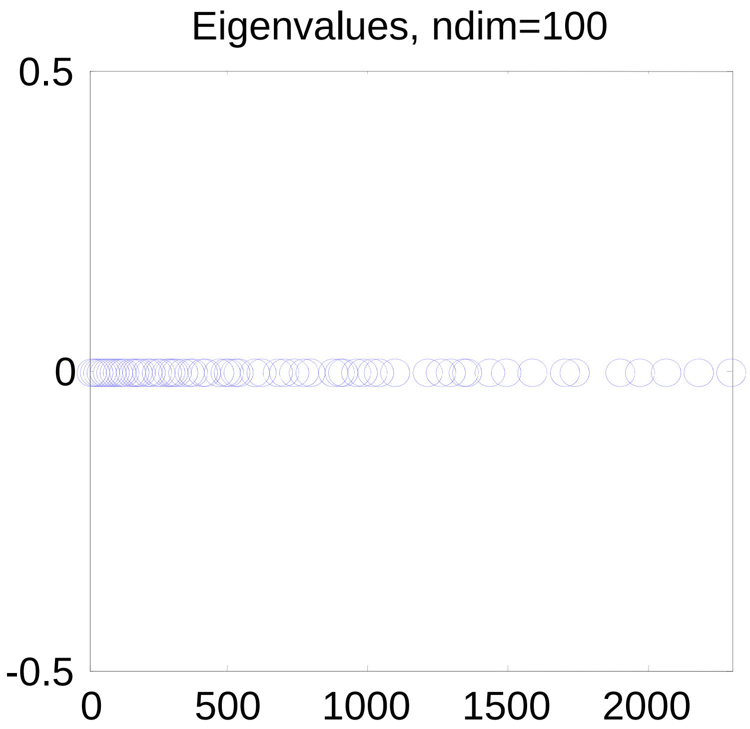
Well-conditioned matrix

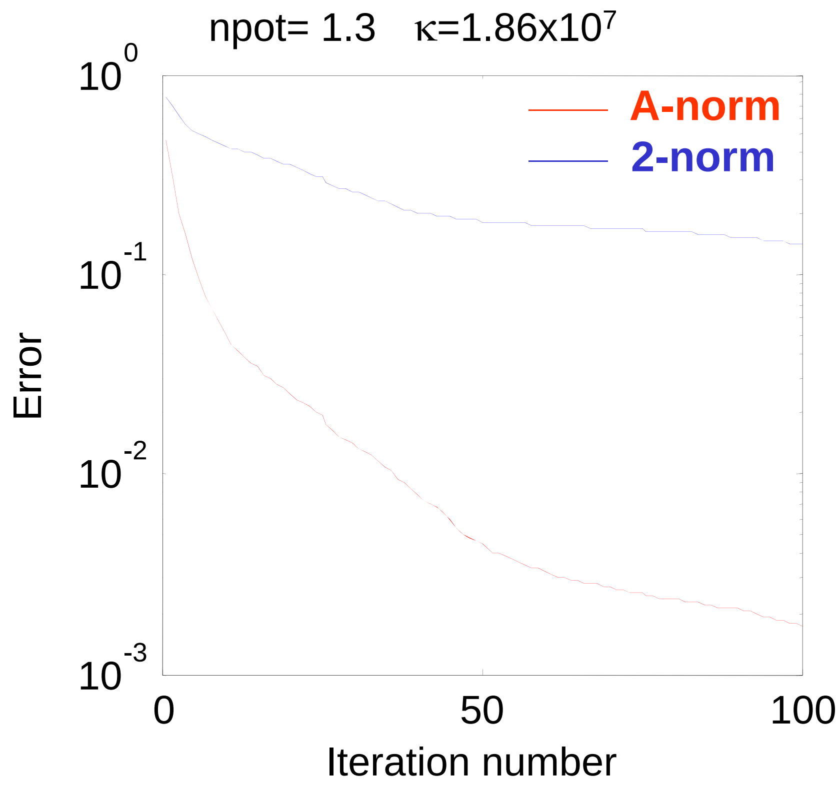


Medium Conditioned Matrix



Ill-conditioned Matrix





ERROR ANALYSIS

Krylov sequence

$$x_k - x_0 \in \text{span}\{g_0, Ag_0, A^2 g_0, \cdots, A^{k-1} g_0\}$$

$\Rightarrow$

Matrix polynomial

$$x_k - x_0 = \sum_{j=0}^{k-1} \gamma_j A^j g_0 = P_k(A) g_0 = P_k(A)(Ax_0 - b)$$

&

$$x_k - x^* = (x_k - x_0) + (x_0 - x^*)$$

$\Rightarrow$

$$x_k - x^* = [P_k(A) A + I](x_0 - x^*) = Q_k(A)(x_0 - x^*)$$

$\Rightarrow$

$\{\lambda_1, \cdots \lambda_n\}$ *eigenvalues*

$$\|x_k - x^*\|_A^2 = \sum_{i=1}^{n} Q_k^{\,2}(\lambda_i) \lambda_i \xi_i^{\,2}, \quad \|x_0 - x^*\|_A^2 = \sum_{i=1}^{n} \lambda_i \xi_i^{\,2}$$

THE EXACT ERROR BOUND

$$\left\| x_k - x^* \right\|_A^2 = \sum_{j=1}^{n} Q^2\left(\lambda_j\right)\xi_j^2,$$

$$x_0 - x^* = \sum_{j=1}^{n} \xi_j e_j,$$

where, because we know that this A-norm is as small as possible:

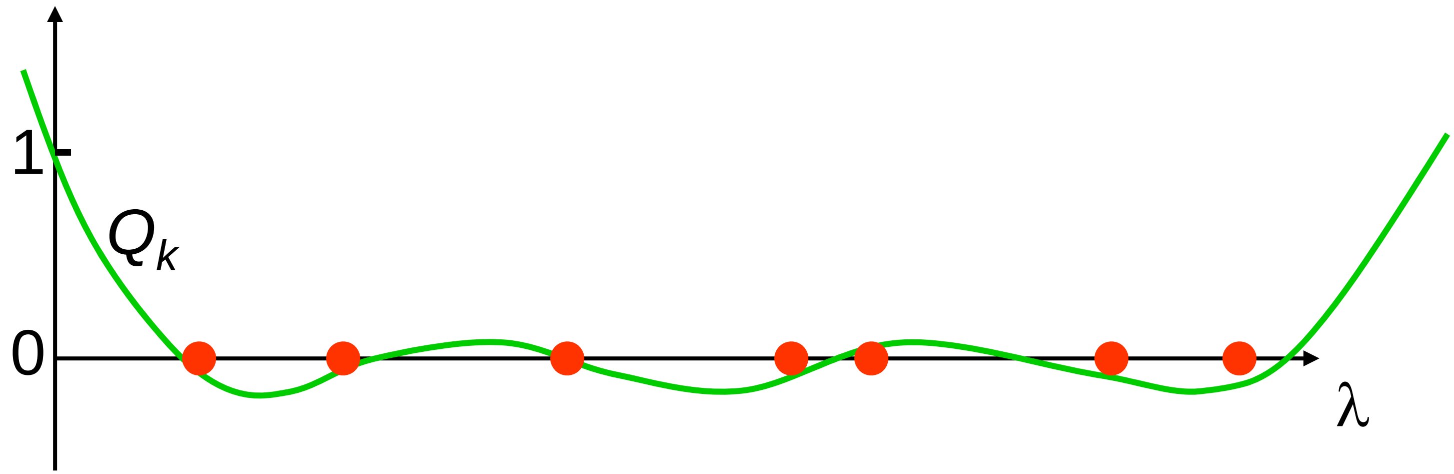
$$Q = \underset{\substack{Q \text{ k-th order,} \\ Q(0)=1}}{\arg\min} \left(\sum_{j=1}^{n} Q^2\left(\lambda_j\right)\xi_j^2, \right)$$

A more convenient form:

$$\frac{\left\| x_k - x^* \right\|_A}{\left\| x_0 - x^* \right\|_A} \leq \max_i \left| Q_k \left(\lambda_i \right) \right|$$

Q_k of order k

$Q_k(0) = 1$



● *n eigenvalues (some may be equal)*

For the *best universal bound* on the convergence of the CG-method (if we only know the extreme eigenvalues), we need to solve

$$Q(\lambda) = \underset{\substack{Q \text{ k-th order,} \\ Q(0)=1}}{\arg\min} \left(\max_{\lambda_1 \le \lambda \le \lambda_n} \left| Q(\lambda) \right| \right)$$

Solution:

$$Q(\lambda) = \frac{T_k\left(\dfrac{2\lambda - \lambda_1 - \lambda_n}{\lambda_n - \lambda_1}\right)}{T_k\left(\dfrac{\lambda_1 + \lambda_n}{\lambda_1 - \lambda_n}\right)}$$

The Chebyshev Polynomials –
the flattest polynomials in the universe!

$$T_k\left(x\right) = \cos\left(k \arccos x\right)$$

$$T_0\left(x\right) = 1$$

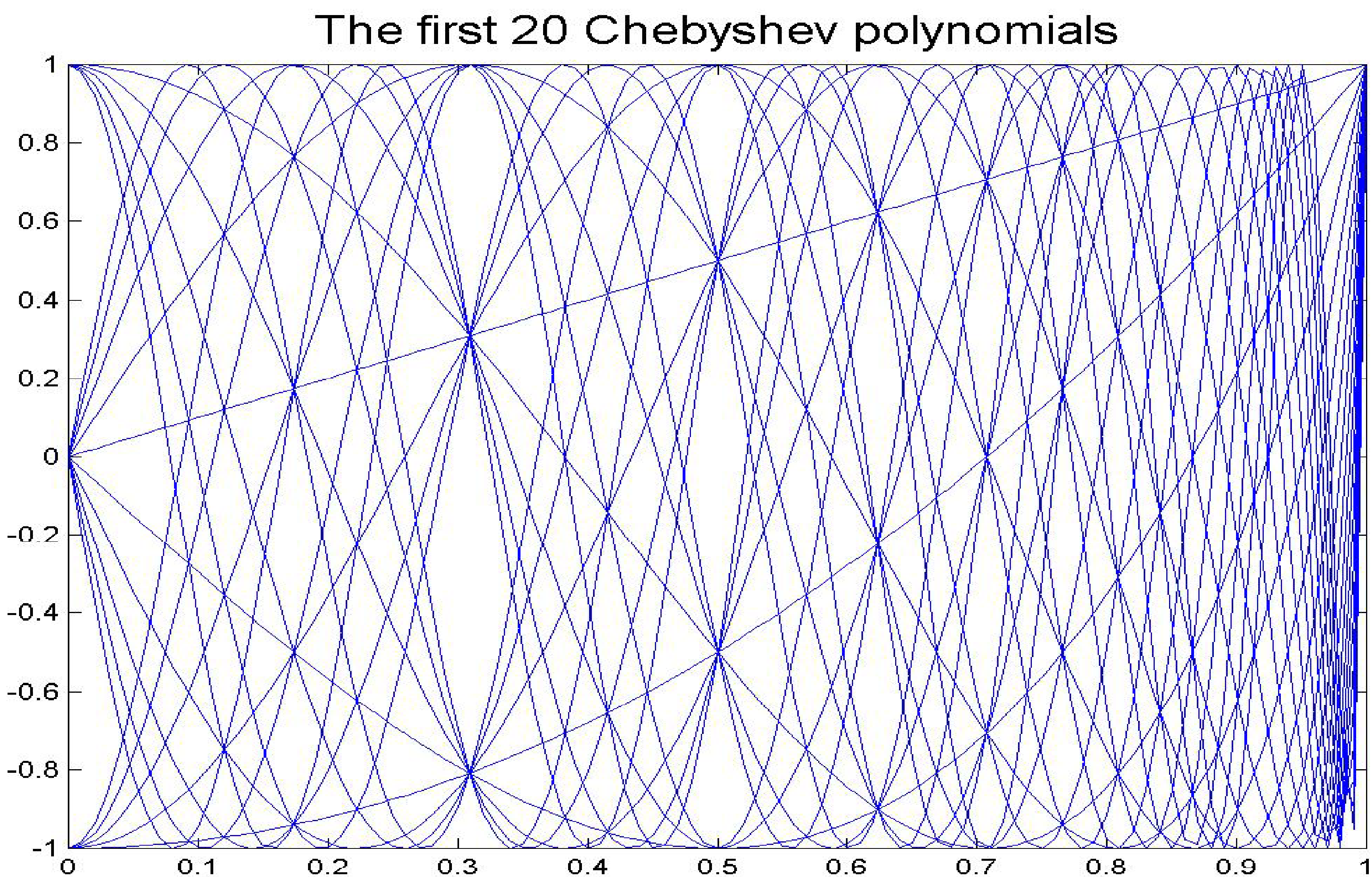
$$T_1\left(x\right) = x$$

$$T_2\left(x\right) = -x + x^2$$

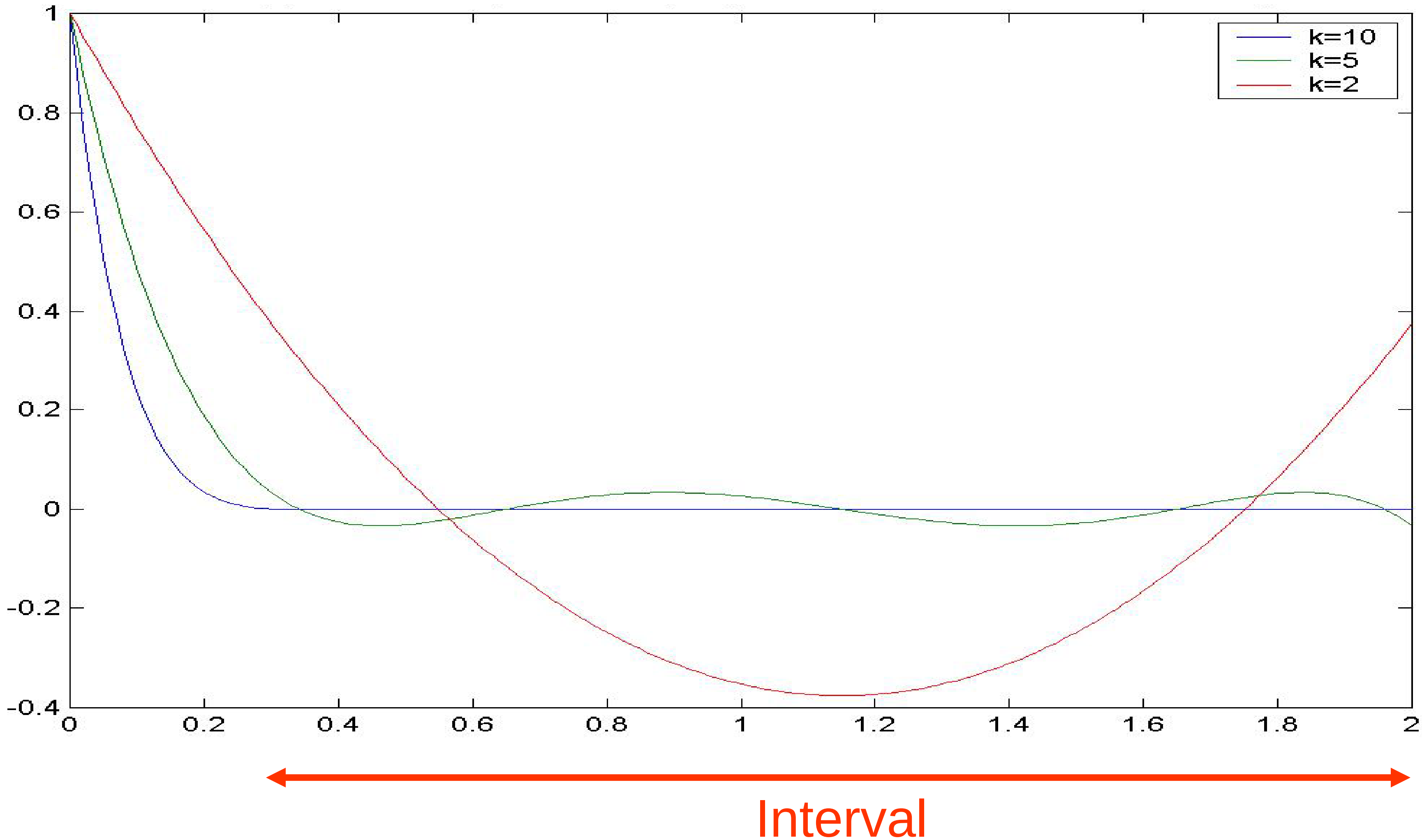
$$T_3\left(x\right) = -3x + 4x^3$$

$$T_4\left(x\right) = 1 - 8x^2 + 8x^4$$

(interval [-1,1])


The first 20 Chebyshev polynomials
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

The optimal Ch. polynomials on the interval [0.3, 2.0]



$$\max_{\lambda_1 \le \lambda \le \lambda_n} \left| Q(\lambda) \right| = \left| \frac{T_k\left(\frac{2\lambda - \lambda_1 - \lambda_n}{\lambda_n - \lambda_1} \right)}{T_k\left(\frac{\lambda_1 + \lambda_n}{\lambda_1 - \lambda_n} \right)} \right| = \frac{1}{T_k\left(\frac{\kappa + 1}{\kappa - 1} \right)} \le 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^k$$

$$\left\| x_k - x^* \right\|_A \le 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^k \left\| x_0 - x^* \right\|_A$$

(more info in the note on the web)