TMA 4180 Optimeringsteori

2004/Rev. 2007

Preparation for the exam - a study in optimal control

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- By using *variational calculus* it is possible to formulate and analyze a simple mathematical model of how one should prepare for an exam.

- Since the optimal solution appears to reproduce the typical student preparation, it is not necessary optimal for *long term* knowledge!

1 The Model

- Learning depends on the *effort*, $I^*(t^*)$.

- Effort below a certain level I_0 gives no significant contribution to the insight/knowledge!

- Very high effort is not necessary very efficient!

Acquired knowledge per time unit as a function of the effort:

$$D\left(I^*\right) = \kappa \ln^+ \left(\frac{I^*}{I_0}\right)$$

$$\ln^+(x) = \begin{cases} 0, & x \leq 1, \\ \ln x, & x > 1. \end{cases}$$

The *learning ability coefficient*, κ

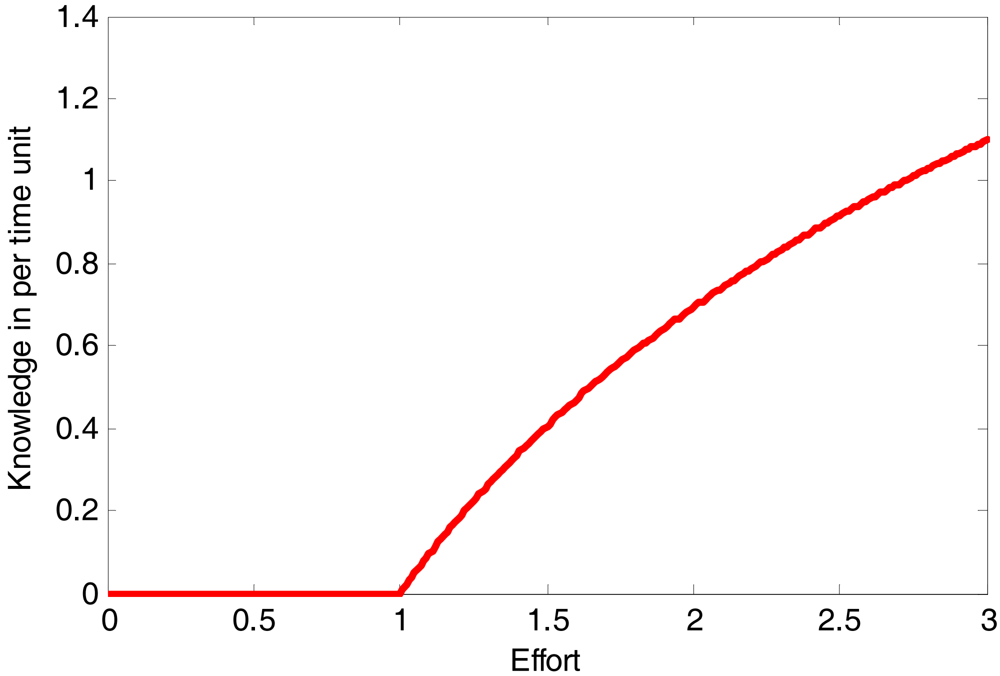

Figure 1: Acquired knowledge per time unit as a function of the effort.

Acquired knowledge at time t^*: $L^*(t^*)$

The *breakdown* of knowledge:

$$\frac{dL^*(t^*)}{dt^*} = -\frac{L^*(t^*)}{\alpha},$$

Breakdown time constant: α.

Dynamic model for the knowledge:

$$\frac{dL^*(t^*)}{dt^*} = \kappa \ln^+ \left(\frac{I^*}{I_0} \right) - \frac{L^*(t^*)}{\alpha}$$

- No use in an effort I^* less than I_0!

- The students start without any knowledge about the subject: $L^*(0) = 0$.

At the time of the exam (at time $t^* = T$), the *target* is $L^*(T) = p$, that is, the *curriculum*.

AIM: *How do we reach the target with the minimal total effort*,

$$J^*(I^*) = \int_0^T I^*(t^*) dt^*.$$

2 Scaling and Problem Formulation

A *scale* is a natural measuring stick for a variable (more about this in the Mathematical modelling course).

Reasonable (but not the only) scales:

$$\begin{aligned} L^* &= pL, \\ I^* &= I_0 I, \\ t^* &= \alpha t, \\ T &= \alpha a. \end{aligned}$$

The differential equation in dimensionless form:

$$\frac{dL(t)}{dt} = \frac{1}{\mu} \ln^+ I(t) - L(t),$$

Dimensionless curriculum: $\mu = \frac{p}{\alpha\kappa}$

Dimensionless reading period: a

The expression for total effort:

$$J^*(I^*) = \alpha I_0 \int_0^a I(t)dt,$$

We therefore consider

$$J(I) = \int_0^a I(t)dt.$$

PROBLEM:

$$\min_I J(I) = \min_I \int_0^a I(t)dt,$$

when

$$\frac{dL(t)}{dt} = \frac{1}{\mu} \ln^+ I(t) - L(t),$$
$$L(0) = 0,$$
$$L(a) = 1.$$

How do we solve it??

The general solution of the differential equation:

$$L(t) = Ce^{-t} + \frac{1}{\mu}\int_{\tau=0}^{t} \ln^{+}(I(\tau))\, e^{-(t-\tau)} d\tau.$$

Since $L(0) = 0$, $C = 0$.

FINAL FORMULATION:

$$\min \int_0^a I(t)dt,$$

when

$$G(I) = \int_0^a \ln^{+}(I(t))\, e^{t-a} dt = \mu.$$

3 Solution and Analysis

- Both functionals have the *standard form*

- The functional J is *convex*

- Since $d^2 \ln^+(x)/dx^2 < 0$ for $x > 1$, and $e^{t-a} > 0$, then the kernel $\ln^+(I(t))\, e^{t-a}$ is *strongly concave*. Thus, $-G(I)$ is strictly convex when $I > 1$.

- The Lagrange functional
$$\mathcal{L}(I) = J(I) - \lambda G(I),$$
is strictly convex for $\lambda > 0$, as long as $I(t) \geq 1$

- Any solution of $\delta\mathcal{L}(I, v) = 0$ is *then* a global minimum (Troutman, Theorem 3.16).

COMPLICATION: If μ is too small, it does not pay to read the whole period a!

3.1 Case A: $I(t) > 1$ for all $t \in [0\,, a]$.

$$\mathcal{L}(I) = \int_0^a \left(I(t) - \lambda \ln \left[I(t) \right] e^{t-a} \right) dt.$$

The Euler equation:

$$\frac{\partial}{\partial I} \left[I - \lambda \left(\ln I \right) e^{t-a} \right] = 1 - \frac{\lambda}{I} e^{t-a} = 0.$$

Solution:

$$I\left(t\right) = \lambda e^{t-a}.$$

The constant λ is found from

$$\begin{aligned} \mu &= \int_0^a \left(\ln \lambda + (t-a) \right) e^{t-a} dt \\ &= \ln \lambda \cdot \left(1 - e^{-a} \right) + (a+1) e^{-a} - 1, \end{aligned}$$

or

$$\ln \lambda = \frac{\mu e^a + e^a - a - 1}{e^a - 1}.$$

This is an optimal solution if

$$I\left(t\right) = \lambda e^{t-a} \geq 1 \text{ for all } t \in [0, a]\,,$$

that is

$$\lambda e^{-a} \geq 1,$$

or

$$\ln \lambda \geq a.$$

This holds when

$$\mu \geq a - 1 + e^{a}.$$

Let a_0 be the solution of

$$\mu = a_0 - 1 + e^{a_0}$$

(a_0 is the limiting length of the reading period for this case)

If $a \leq a_0$, then the optimal effort function, assuming that we choose to read the whole period a, is

$$I_{opt}(t) = \frac{\lambda}{e^{a}} e^{t}, \ \lambda = \exp\left(\frac{\mu e^{a} + e^{a} - a - 1}{e^{a} - 1}\right)$$

An exponential increase towards the exam!

The differential equation is

$$\frac{dL(t)}{dt} + L(t) = \frac{1}{\mu} \ln\left(\frac{\lambda}{e^a} e^t\right) = \frac{\ln\lambda - a + t}{\mu}$$

with solution

$$L(t) = \frac{\ln\lambda - a - 1 + t}{\mu} - \frac{\ln\lambda - a - 1}{\mu} e^{-t}$$

(Yes! $L(0) = 0$ and $L(a) = 1$!)

3.2 Case B: $a > a_0$

- $\mu < a - 1 + e^a$

- $I(t) = \lambda e^{t-a} < 1$ in part of the interval

- The Lagrangian, $\mathcal{L}(I) = J(I) - \lambda G(I)$ is *no longer convex* (cf. Fig. 1)

Smart idea: *Do not start the reading until the time to the exam is a_0. Then follow the optimal solution above.*

Thus, do not start too early because you then forget too much during the reading period!

Smarter idea: *What is the optimal length of the reading period a among all with $a \leq a_0$?* (For each such a, we use the optimal strategy)

Effort as a function of a:

$$\begin{aligned} J\left(a\right) &= \int_0^a I(t)dt \\ &= \int_0^a \lambda e^{t-a}dt \\ &= \lambda(1-e^{-a}). \end{aligned}$$

Thus,

$$\begin{aligned} \ln J &= \ln\lambda + \ln(1-e^{-a}) \\ &= \frac{\mu e^a + e^a - a - 1}{e^a - 1} + \ln(1-e^{-a}) \\ &= 1 + \frac{\mu e^a - a}{e^a - 1} + \ln(1-e^{-a}) \end{aligned}$$

$$\frac{d\ln J}{da} = e^a \frac{a-\mu}{(e^a-1)^2},$$

which has a minimum for $a = \mu$.

For the optimal situation $a = \mu$:

$$\begin{aligned} I(t) &= e^{t+1}, \\ L(t) &= \frac{t}{a}. \end{aligned} \tag{1}$$

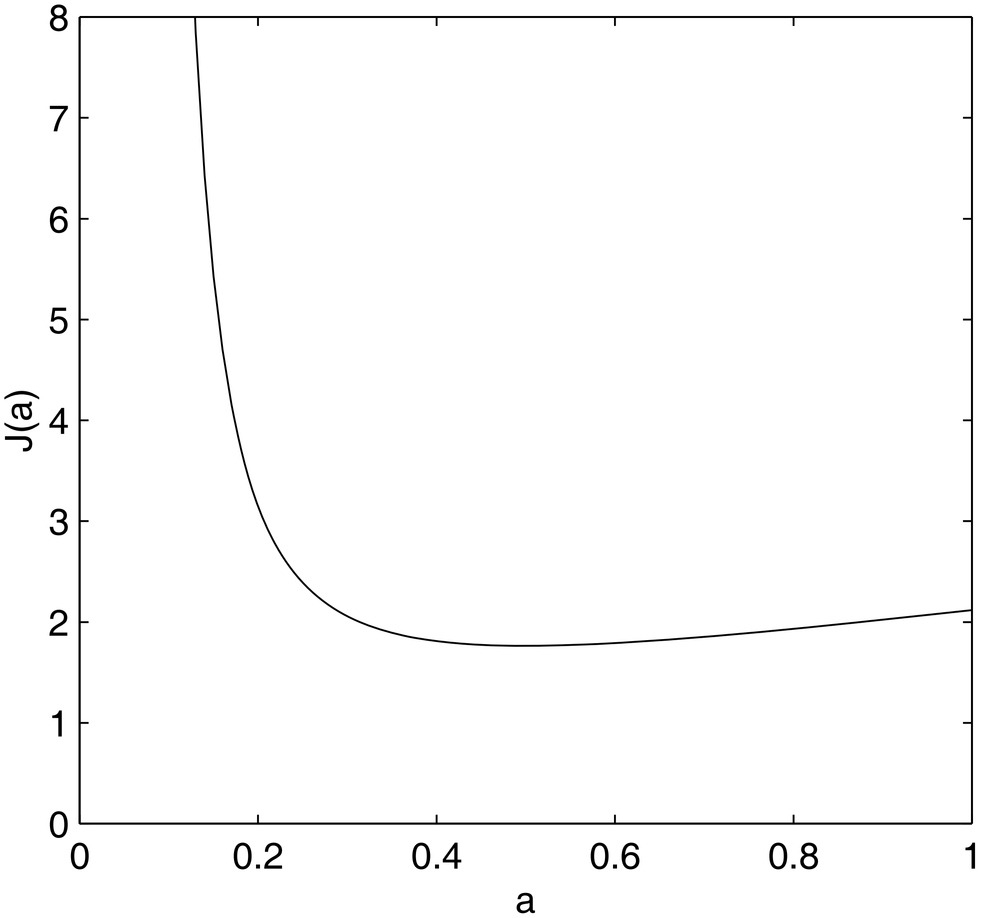

Figure 2: Total effort in order to learn the curriculum as a function of the length of the reading period ($\mu = 0.5a_0$).

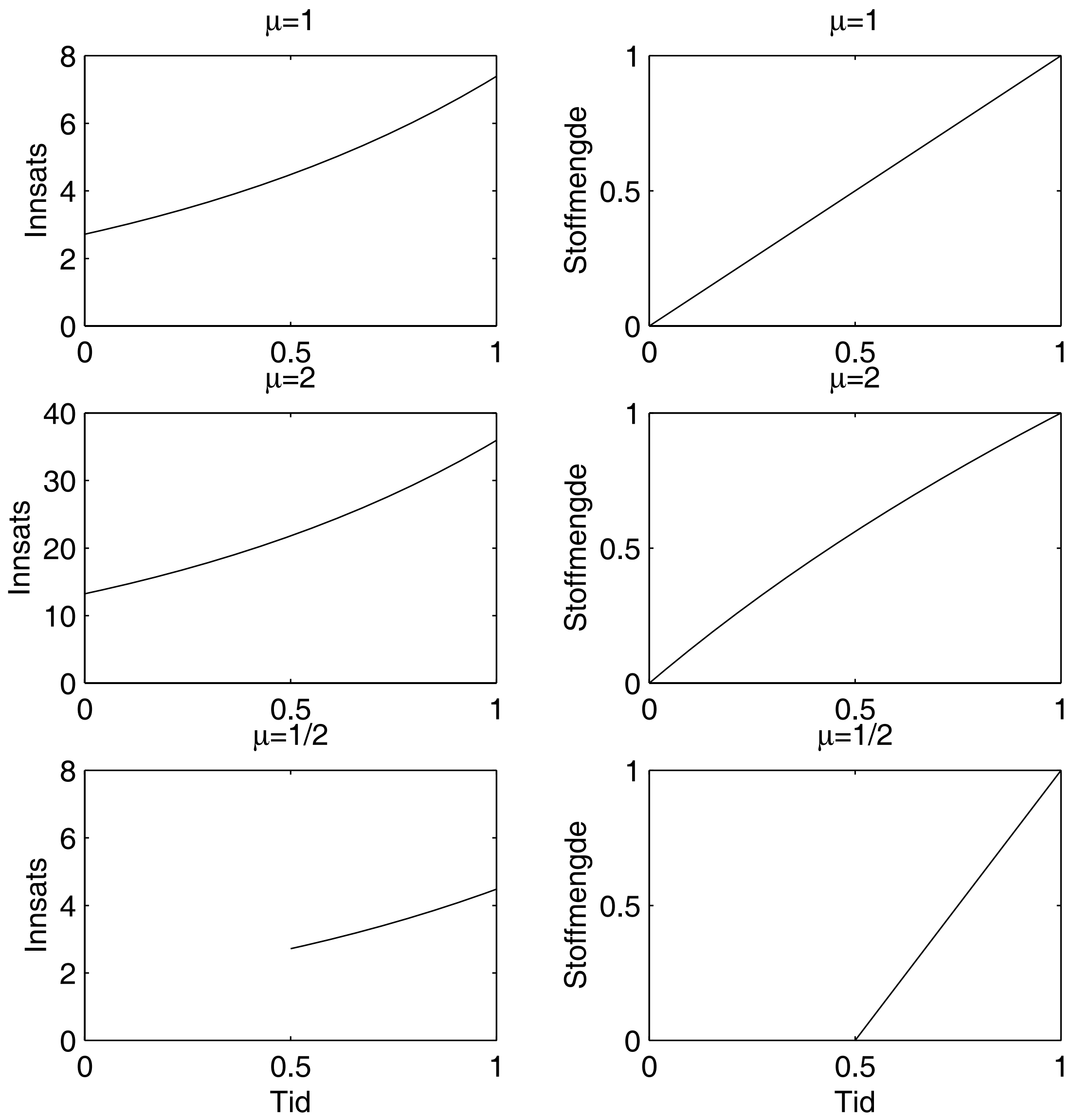

Figure 3: Some other optimal cases (*Tid* = Time, *Innsats* = Effort, *Stoffmengde* = Curriculum).

4 Summary

Parameters:

Length of the semester: T

Low effort limit: I_0

The learning ability coefficient: κ

Forgetfulness time constant: α

The curriculum: p

Dimensionless parameters:

Dimensionless curriculum: $\mu = \frac{p}{\kappa\alpha}$

The dimensionless limiting period a_0 (solution of $\mu = a - 1 + e^a$)

The optimal starting time given by $a_{opt} = \mu$

Overall optimal starting time ($\mu = a_{opt}$) given by $\frac{p}{\kappa\alpha} = a_{opt}$, or

$$T_{opt} = \frac{p}{\kappa}.$$

Define also

$$T_0 = \alpha a_0$$

5 Conclusions

1. Read continuously!

2. If $T_{opt} \leq T$, do not start the reading before T_{opt} and then follow the optimal strategy, $I(t) = e^{t+1}$, that is,

$$I_{opt}(t^*) = I_0 \exp\left(\frac{t^*}{\alpha} + 1\right).$$

3. If $T_{opt} \geq T$, start at once and follow the strategy

$$I_{opt}(t^*) = I_0 \lambda \exp\left(\frac{t^*}{\alpha} - \frac{T}{\alpha}\right),$$

$$\lambda = \exp\left(\frac{\mu e^a + e^a - a - 1}{e^a - 1}\right).$$

4. Never start before T_0!

5. It is reasonable to start so that

$$T_{opt} \leq T_{start} \leq \min(T, T_0)$$

6. Do not wait until $T_{start} < T_{opt}$!

7. For the teacher: By keeping the curriculum large enough, the students will find that they have to study the whole term with an exponential increase towards the exam!