Optimisation of Production

(Troutman, pp. 72–74)

- $L(t)$: Inventory ($=$ amount in storage)

- $P(t)$: Production per time unit

- $S(t)$: Sales per time unit

Equation for the inventory (*system dynamics*):

$$\frac{dL(t)}{dt} = P(t) - S(t) - \alpha L(t) .$$

Here, $-\alpha L$ is loss (destruction) at the storage per time unit.

- $\mathcal{S}(t)$: Sales prognosis (*Expected* sales)

Optimal inventory ($\mathcal{L}$) and production ($\mathcal{P}$) are matching the sales prognosis $\mathcal{S}$,

$$\frac{d\mathcal{L}(t)}{dt} = \mathcal{P}(t) - \mathcal{S}(t) - \alpha\mathcal{L}(t)\,.$$

E.g., if we want $\mathcal{L}(t)$ to be constant, we need a production $\mathcal{P}(t) = \mathcal{S}(t) + \alpha\mathcal{L}(t)\,.$

At $t = 0$, the inventory is off the ideal inventory, $L(0) = L_0 \neq \mathcal{L}(0)$.

How do we plan the production $P(t)$ so as to minimize the extra cost of being off the ideal situation?

The Cost Functional

(what we suffer from not being on the ideal track)

- $L(t) - \mathcal{L}(t)$: Deviation from optimal inventory
- $P(t) - \mathcal{P}(t)$: Deviation from optimal production

Common formulation:

$$C(P) = \int_0^T \left[\beta^2 (L(t) - \mathcal{L}(t))^2 + (P(t) - \mathcal{P}(t))^2\right] dt$$

This is an *Optimal Control problem*:

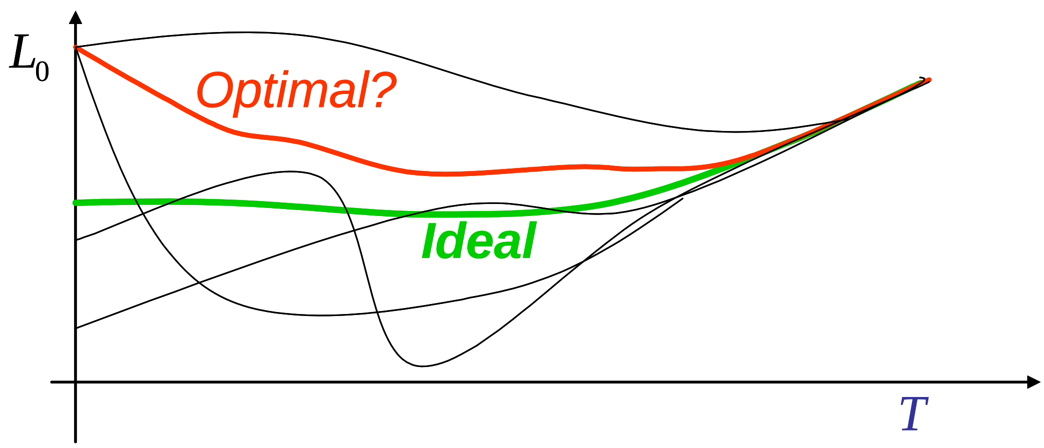
$$\min_P C$$
$$L'(t) = P(t) - \mathcal{S}(t) - \alpha L(t),$$
$$L(0) = L_0.$$

- The variable $P(t)$ is the *control variable*

- We expect the sales prognosis $\mathcal{S}(t)$ to be true, and try to adjust the inventory towards the optimal $\mathcal{L}(t)$

- Inventory is forced to follow the system dynamics:

$$\frac{dL(t)}{dt} = P(t) - \mathcal{S}(t) - \alpha L(t), \; L(0) = L_0.$$



We are starting off the ideal curve (which we *think* we know), and want to minimize the cost of getting there.

Observe that P may be expressed as

$$P = L' + \alpha L + \mathcal{S}.$$

Let $y = L(t) - \mathcal{L}(t)$. Then, since

$$L' = P - \mathcal{S} - \alpha L,$$
$$\mathcal{L}' = \mathcal{P} - \mathcal{S} - \alpha \mathcal{L},$$

we have

$$L' - \mathcal{L}' = P - \mathcal{P} - \alpha (L - \mathcal{L}),$$

or

$$P - \mathcal{P} = \frac{dy}{dt} + \alpha y.$$

This is inserted into the cost functional:

$$C(y) = \int_0^T \left[\beta^2 y^2 + \left(y' + \alpha y\right)^2\right] dt$$

- $\beta^2 y^2$ is strongly convex and $(z + \alpha y)^2$ is convex

Thus, C is *strictly convex*.

Solution

The optimal solution is found by solving the Euler equation for C:

$$\begin{aligned}\frac{d}{dx}f_{y'} - f_y &= \frac{d}{dx}\left[2\left(y' + \alpha y\right)\right] - \left[2y\beta^2 + 2\left(y' + \alpha y\right)\alpha\right] \\ &= 2\left(y'' - \left(\alpha^2 + \beta^2\right)y\right) = 0.\end{aligned}$$

with a fixed value at $t = 0$,

$$y(0) = y_0 = L_0 - \mathcal{L}_0.$$

and a natural condition at $t = T$ (we have no particular restriction on y at T):

$$f_{y'}(y(T)) = 2\left(y'(T) + \alpha y(T)\right) = 0.$$

With $\gamma^2 = \left(\alpha^2 + \beta^2\right)$, the problem may now be formulated as

$$\begin{aligned}y'' - \gamma^2 y &= 0, \\ y(0) &= y_0, \\ y'(T) + \alpha y(T) &= 0.\end{aligned}$$

From the general solution of the Euler equation

$$y(t) = A \exp(\gamma t) + B \exp(-\gamma t),$$

it is easy to find the unique optimal solution

$$y^*(t)) = y_0 \frac{e^{\gamma t} + \rho e^{-\gamma t}}{1+\rho}, \quad \rho = \frac{\gamma + \alpha}{\gamma - \alpha} e^{2\gamma T}.$$

The minimal value for the functional may also be computed:

$$C(y^*) = y_0^2 \left[\frac{\gamma(\rho - 1)}{\rho + 1} - \alpha \right].$$

Application: The Ice Cream Factory

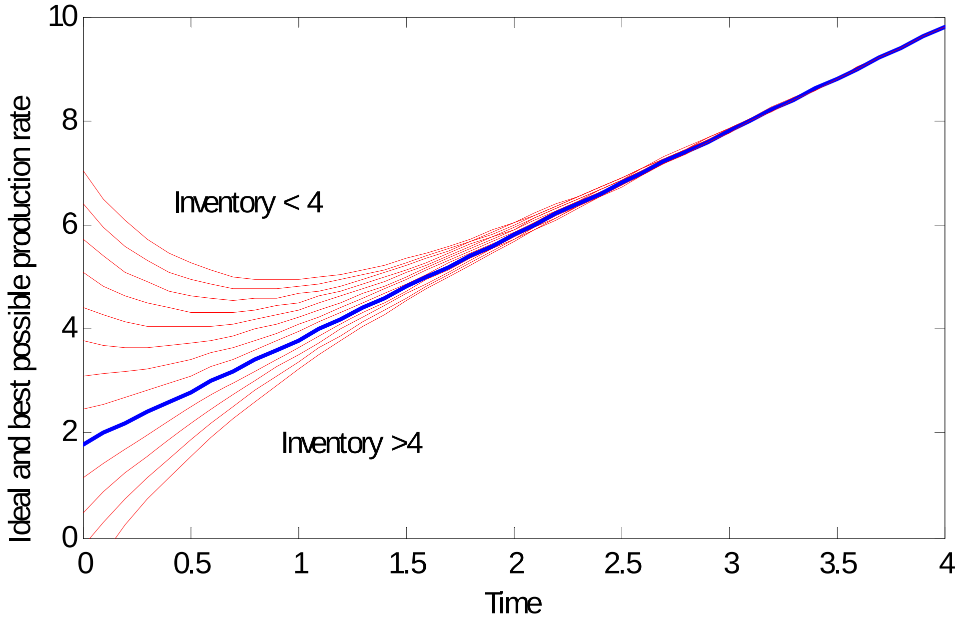
The factory is planning for the summer. Today, $t = 0$, is April 19th, and $t = 4$ is August 19th.

- Sales prognosis: $\mathcal{S}(t) = 1 + t$

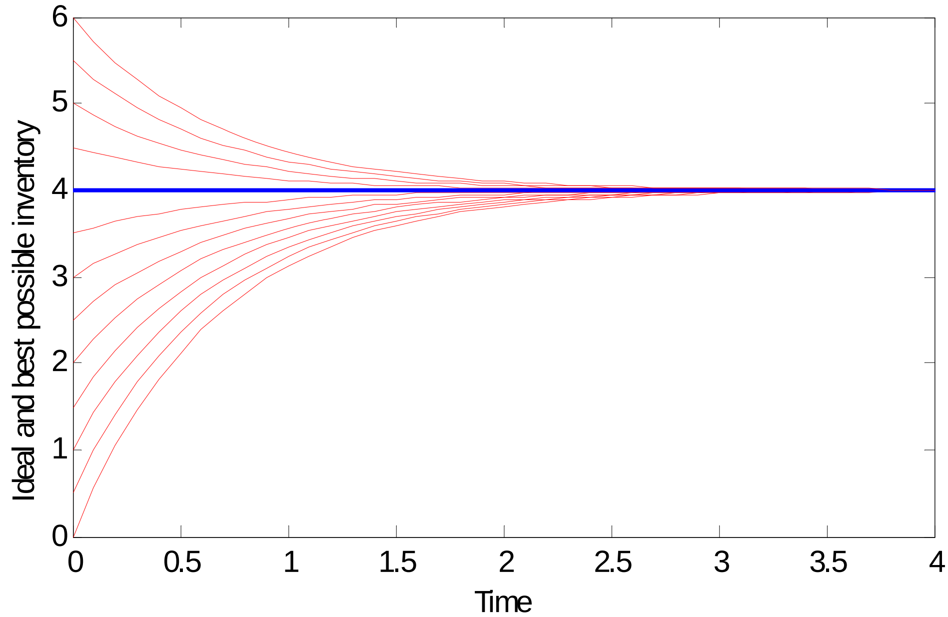
- Optimal inventory: $\mathcal{L}(t) = 4$

- The optimal production rate for $\mathcal{L}(t) = 4$ is $\mathcal{P}(t) = \mathcal{S}(t) + \alpha \times 4$.

For a decay rate, $\alpha = 0.1$, and $\beta = 1.5$ in the cost functional, the best production and inventory for various values of L_0 are shown on the plots.



Ideal and best possible production rate
Time
Inventory < 4
Inventory >4



Ideal and best possible inventory
Time