TMA 4180 Optimeringsteori
Linear Programming Basics

H. E. Krogstad

Dept. Mathematical Sciences, 2012.

About this note: Some years ago the basic theory of Linear Programming was set up as a self-study part. In order to ease the work, the note below was written as a guide to Chapter 13 in N&W. I still think that reading the note and carry out all "Tasks" is the best way of absorbing this material! The note covers the LP part of the curriculum.

Linear Programming is first of all a somewhat unfortunate name of what today should have been called *Linear Optimization*. This change in terminology has never occurred, and we shall follow the rest of the bunch and continue to use Linear Programming, or "*LP*" for short.

Write in "Linear Programming" or "LP optimization" as search words in *Google*, and see that the world is full of consulting companies specializing in LP services!

Linear programming means

- *a linear objective function!*
- *linear constraints!*

A **Task** below is a small piece of work to be carried out by **you**!

1 The LP Problem

We are now considering problems where the objective function is *linear*

$$f(x) = a + c'x. \tag{1}$$

Since a is a constant, we shall always assume that $a = 0$ and $c \neq 0$. Note that since $\nabla f = c' \neq 0$, all solutions have to be on the boundaries of the *feasibility domain,* Ω, which in general is defined in terms of linear *equality constraints*,

$$a_i' x = b_i, \ i \in \mathcal{E}, \tag{2}$$

and linear *inequality constraints*,

$$a_i' x \geq b_i, \ i \in \mathcal{I}. \tag{3}$$

In addition, since second derivatives will be 0, there are no second order conditions to consider in the LP-case.

Task 1: *Show that the objective function and the feasible domains are convex. What geometric sets are defined by Eq. 2 and 3?*

This implies that if solutions exist, they are global minima and make up a convex set. By the Convex KKT theorem (see the KKT note), a KKT point is a therefore a global minimum. Moreover, the LICQ condition is not necessary. In conclusion, for the LP-situation, x^* is a global minimum if and only if it is a KKT-point (This does not imply that x^* is unique!).

Task 2: *There are two situations where no solution exists, and one is that the constraints are incompatible such that $\Omega = \varnothing$. What other situation may occur ? Hint: Is Ω always bounded?*

The feasible domain Ω, as an intersection of hyperplanes and half-spaces, is called a *polytope*. A bounded polytope is called a *polyhedron*.

It is convenient to have a common form for the LP problems when we go through the theory, and it is customary to call the following problem the *Standard Form*:

$$\begin{gathered} \min \; c'x \\ Ax = b, \; A \text{ has full row rank}, \\ x \geq 0. \end{gathered} \tag{4}$$

We shall write $\mathbb{R}^n_+ = \{x \; ; \; x \geq 0\}$. As usual, "$x \geq 0$" is a short way to write $x_i \geq 0$, $i = 1, \cdots, n$. This set is called the *closed, non-negative cone* in $\mathbb{R}^n$. Thus,

$$\Omega = \mathbb{R}^n_+ \cap \{x \; ; \; Ax = b\}. \tag{5}$$

Task 3: *Show that we, without changing Ω, may reduce the number of rows in A to a situation where it has full row rank. Show that this reduction may lead us into the following situations*:

- $\Omega = \varnothing$
- $\Omega = \{x^*\}$
- $\Omega = \{x_0 + \mathcal{N}(A)\} \cap \mathbb{R}^n_+$, where $Ax_0 = b, \; x_0 \geq 0$.

All LP problems may be transformed into the standard form, at the expense of introducing more variables (read first the Introduction pages of Chapter 13 in N&W).

Task 4: *Consider the five cases below and reduce them to the Standard Form. In each case, write down the new A, b, and x*!

1. $x \geq d$, that is, $x_i \geq d_i, \; i = 1, \cdots, n$.
2. $Ax \leq b$. Add a new set of variables z, called *slack variables*, such that $Ax + z = b$.
3. $Ax \geq b$. Similar to above, add a vector z of *surplus variables*, such that $Ax = b + z$.
4. No non-negativity bound on some of the x-s, say $x_f = (x_1, \cdots, x_k)$, $k \leq n$ (we say that x_f is "free"). Write x_f as the difference between the non-negative vectors y and z, $x_f = y - z$.
5. $a \leq x \leq b$, that is, $a_i \leq x_i \leq b_i$, $i = 1, \cdots, n$.

Task 5: *Transform the following problem to standard form:*

$$\begin{gathered} \max\left(-x_1 + x_2\right), \\ x_1 \leq 2 + x_2, \\ 6 - x_2 \geq x_1 \end{gathered} \tag{6}$$

Task 6: *Look up* Matlab Optimization Toolbox *and check whether the present version still uses its own and different "Standard Form":*

$$A_{ieq}x \leq b_{ieq},$$
$$A_{eq}x = b_{eq}, \tag{7}$$
$$v_l \leq x \leq v_h, \tag{8}$$

where

$$-\infty \leq v_l \leq v_h \leq \infty \tag{9}$$

(to be understood component-wise). How can we move between the two "standard forms"?

2 The KKT Conditions

An essential part of the proof of the KKT Theorem was that the LICQ condition should hold. By Lemma A, this implied that the two sets

$$\mathcal{T}(x) = \{d;\ d \text{ is a feasible direction}\}, \tag{10}$$
$$\mathcal{F}(x) = \{d\ ;\ \nabla c_i(x)\, d = 0,\ i \in \mathcal{E}, \nabla c_i(x)\, d \geq 0, i \in \mathcal{I}\}, \tag{11}$$

were equal. For the LP situation $\mathcal{T}(x)$ is always equal to $\mathcal{F}(x)$ (N&W Ed. 2: Lemma 12.7 pp. 338–339.):

Task 7: *Try to see this yourself by first selecting a* $d \in \mathcal{F}(x)$, $x \in \Omega$, *and then prove* d *is feasible by verifying that* $x + td \in \Omega$ *for all sufficiently small and positive* t*-s. Hint: Check that all constraints are valid, both active and non-active when* t *is small enough, and remember that we always have* $\mathcal{T}(x) \subset \mathcal{F}(x)$*).*

Let us consider the KKT-conditions for the standard form defined in Eq. 4. We follow N&W and split the Lagrange vector into two parts; π for the equality constraints and s for the inequality constraints:

$$\mathcal{L}(x, \pi, s) = c'x - \pi'(Ax - b) - s'x. \tag{12}$$

Task 8: *Verify the KKT-equations*

$$c - A'\pi - s = 0, \tag{13}$$
$$Ax = b, \tag{14}$$
$$x \geq 0, \tag{15}$$
$$s \geq 0, \tag{16}$$
$$s'x = 0. \tag{17}$$

Note that because of Eq. 15 and 16, Eq. 17 is equivalent to

$$s_i x_i = 0,\ i = 1, \cdots, n. \tag{18}$$

The KKT-equations are a nonlinear system of equations and inequalities in $2n + \operatorname{rank}(A)$ unknowns. The system looks innocent, but it is often a non-trivial task to find even a single $x \in \Omega$!

3 Duality

Duality is essential for LP-problems, and the concept may also be defined for some nonlinear problems. In Duality Theory, variables and Lagrange multipliers switch place!

Definition: *For every LP problem there is another* ***dual problem*** *with the same KKT-equations, where the Lagrangian multipliers are the unknowns, and the original variables are the Lagrangian multipliers.*

Proposition: *The dual problem to the Standard Problem (Eq. 4) is*

$$\begin{aligned} &\max b'\pi \\ &A'\pi \leq c, \end{aligned} \tag{19}$$

By writing the dual problem as

$$\begin{aligned} &\min\left(-b'\pi\right), \\ &c - A'\pi \geq 0 \end{aligned} \tag{20}$$

and using x as the Lagrange multipliers, the Lagrangian becomes

$$\mathcal{L}\left(\pi, x\right) = -b'\pi - x'\left(c - A'\pi\right), \tag{21}$$

In this case, the vector s is not explicitly present, and KKT equations are

$$\begin{aligned} \nabla_\pi \mathcal{L}\left(\pi, x\right) &= -b' + x'A', \\ c - A'\pi &\geq 0 \text{ (constraints)}, \\ x &\geq 0 \text{ (non-negative multipliers)}, \\ x'\left(c - A'\pi\right) &= 0 \text{ (complementarity equations)}. \end{aligned} \tag{22}$$

However, by introducing $s = c - A'\pi$, we obtain exactly the same KKT-equations as for the standard form:

$$s = c - A'\pi,$$

$$Ax = b, \tag{23}$$

$$x \geq 0,\ \ s \geq 0, \tag{24}$$

$$x's = 0.$$

In duality terminology we use the generic name *Primal* (problem) about our original problem, and *Dual* (problem) about its dual.

There are several ways to find the dual problem of a given primal problem. One way is just by inspection of the KKT-equations, but the simplest is probably to convert any primal to the Standard Form defined in Eq. 4, and then apply the proposition for its dual, Eq. 19. In practice, it seems to be no way around trying to remember this, or the following equivalent variant:

Task 9: *Verify the so-called* **symmetric pair** *of LP primal-dual problems:*

$$\begin{array}{cc} \text{Primal} & \text{Dual} \\ \min c'x & \max b'\lambda \\ Ax \geq b, & c \geq A'\lambda \\ x \geq 0 & \lambda \geq 0 \end{array} \tag{25}$$

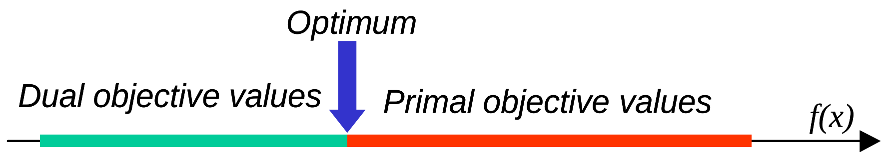

Figure 1: The objective values for the dual are to the left of the values for the primal. The maximum of the dual and the minimum of the primal objectives are the same.

This form may alternatively be memorized and used for identifying dual (or primal) problems.

The following task is one of the basic results in Duality Theory:

Task 10: *Verify, by applying the result in Task 9, that the "dual problem of a dual problem is the primal problem" (after suitable reformulations).*

Apart from an interesting correspondence, and a certain "duality" in diet-food manufacturer problems, duality plays a major part in many algorithms for LP problems. In fact, if we know an approximate solution for our primal problem and also an approximate solution for the corresponding dual, we have an accurate measure of how far the objective is from its optimum value. A close-to-optimal objective and not necessarily an accurate x^* may in many cases be our main goal.

We have seen above that a given LP problem may be expressed in many different, but equivalent forms, and this also applies to the duals. Thus, we follow N&W and consider the primal in Eq. 4 and the dual in Eq. 19.

The Duality Theorem (N&W Theorem 13.1):

1. *If either the primal or the dual problems have feasible points, then so does the other, and the optimum objective values are equal.*
2. *If either problem has an unbounded objective, then the other problem has no feasible points.*

Proof: Assume first that the primal has a finite and hence optimal solution (x^*, π^*, s^*) to the KKT-equations. Then, for any feasible point $x \in \Omega$, and since $c = A'\pi^* - s^*$, s^* and $x^* \geq 0$, and $s^{*\prime}x^* = 0$,

$$c'x \geq c'x^* = \left(\pi^{*\prime}A - s^{*\prime}\right)x^* = \pi^{*\prime}Ax^* = \pi^{*\prime}b. \tag{26}$$

However, since the dual problem is an LP-problem by itself, and in particular convex, the solution (x^*, π^*, s^*) is *also* a solution to the dual (identical) KKT-equations! Thus, π^* will be an optimal solution to the dual problem, and we can turn around and complete the above chain to

$$b'\pi \leq b'\pi^* = c'x^* \leq c'x \tag{27}$$

The situation is illustrated in Fig. 1: All the objective values of the dual problem are *below* the objective values for the primal, and the optimal solutions of both problems have identical values on the objective.

Task 11: *Try to complete the proof of the Duality Theorem, and then look into N&W*, pp. 361–362.

In the *primal-dual methods* of LP (N&W Sec. 14.1) we try to solve the primal and the dual simultaneously. At all steps, the *duality gap*, that is, the interval

$$\left[b'\pi_{(k)}\ ,\ c'x_{(k)}\right], \tag{28}$$

encloses the optimal objective value. This is excellent if a good value on the objective is our main goal.

4 The Geometry of Ω

In order to approach an algorithm for LP-problems, we shall take a closer look at the geometry of the feasible set. This will lead us into some interesting properties of convex sets.

Consider again the standard problem with A having full (row) rank. We have already observed that Ω is a convex polytope. Since the number of constraints are finite, the boundary will be "piecewise flat". If we think of a polyhedron in $\mathbb{R}^3$, the boundary points may be on a face, an edge, or at a corner (*vertex*). These concepts may be generalized, and, in particular, we say that a point x_e on the boundary of Ω is an *extreme point* if

$$x_e = \theta y + (1-\theta)\, z\ ,\ \ y, z \in \Omega\ ,\ \ 0 < \theta < 1 \tag{29}$$

implies that $y = z = x_e$. This means that x_e can not be a point on a line segment in Ω, unless it is one of the end-points.

Task 12: *Where are the extreme points for a line segment, for $\mathbb{R}$ and $\mathbb{R}^n_+$, a cube, and a sphere* (*all sets closed*)?

Digression: The famous *Krein-Milman Theorem* is a cornerstone in convexity theory: *A closed, bounded (more exact, compact) convex set is equal to the closure of all convex combinations of its extreme points* (A convex combination is like the one in Eq. 29, but involving m points, $x = \sum_{i=1}^{m} \theta_i y_i$, with $\theta_i > 0$ and $\sum_{i=1}^{m} \theta_i = 1$).

Task 13: *How does the Krein-Milman Theorem work for a cube*?

We are going to identify the extreme points in Ω as so-called *basic points*, or *basic feasible points*. Write $A = [a_1, a_2, \cdots, a_n]$ where $\{a_i\}$ are the columns of A, and A has full rank, rank $(A) = r$.

Definition: *A feasible point x satisfying $Ax = b$, $x \geq 0$, is called a basic point if there is an index set $\mathcal{B} = \{i_1, \cdots, i_r\}$, where $\{a_{i_1}, \cdots, a_{i_r}\}$ are linearly independent and $x_i = 0$ for all $i \notin \mathcal{B}$.*

This note will be using the term *basic point*, and not the somewhat unfortunate name *basic solution* also found in the LP literature. We shall reserve the term *basic solution* to an *optimal solution which is also a basic point.*

If x_i happens to be 0 also for some $i \in \mathcal{B}$, we say that the basic point is *degenerate.*

Note that for a basic point, the corresponding $r \times r$ matrix

$$B = [a_{i_1}, \cdots, a_{i_r}]\,, \tag{30}$$

will be *non-singular*, and the equation $Bx_B = b$ will have a unique solution.

The basic points in Ω play an essential part in the following. In fact, the next result is central:

The Fundamental Theorem for LP (N&W Theorem 13.2)

1. *If $\Omega \neq \varnothing$, it contains basic points.*

2. *If there are optimal solutions, there are also basic solutions.*

Proof of 1: (Note that for the following arguments, we assume that the problem is stated in standard form, Eq. 4). Find in some way, since $\Omega \neq \varnothing$, an $x \in \Omega$ with as few non-zero components as possible. Assume (by renumbering the variables) that these components are $x_1, x_2, \cdots, x_p$. Hence,

$$b = \sum_{i=1}^{p} a_i x_i. \tag{31}$$

Task 14: *Assume that* $\{a_1, \cdots, a_p\}$ *are linearly dependent and make up a* 0*-combination,* $z = \sum_{i=1}^{p} a_i z_i = 0$*. Add a suitable small amount of* z*,* $y = x + \varepsilon z$*, and obtain a new feasible point (don't forget that* x_i *is supposed to be larger than* 0!),

$$Ay = b,\ y \geq 0, \tag{32}$$

with at most $p-1$ *non-zero components – a contradiction! Having shown that* $\{a_1, \cdots, a_p\}$ *have to be linearly independent, add more linearly independent columns to the set, and show that* x *is a basic point.*

The proof of 2. is similar: *Find an optimal solution* x^* *with as few non-zero components as possible. Assume that* x^* *is not a basic point and use a zero-combination* z *to construct a contradiction: First show that since* x^* *is an optimal solution,* $c'z = 0$*, so that* $y = x + \varepsilon z$ *is optimal as well. Finally, add columns to as to make* x^* *an optimal basic solution.*

The next theorem states the equivalence announced above.

Theorem (N&W Theorem 13.3): *The basic points are the extreme points of* Ω*.*

The proof is similar to the previous one:

Task 15: *Assume that* x *is not a basic point. Show that it is then possible to write*

$$x \ = \frac{1}{2}(x - \varepsilon z) + \frac{1}{2}(x + \varepsilon z), \tag{33}$$

with both $(x - \varepsilon z)$ *and* $(x + \varepsilon z)$ *in* Ω*, so that* x *is not extreme.*

For the converse, assume that x *is a basic point with (after renumbering) at most the first* r *components are different from 0, and* $B = [a_1, a_2, \cdots, a_r]$*. Write* $A = [B, N]$*. Assume that* $x = \theta y + (1-\theta) z$*. Show that* $y = z = x$*, by proving that* y *and* z *are basic solutions as well.*

Do not struggle too much with these tasks. Consult N&W if you do not see a way out!

The Fundamental Theorem and Theorem 13.3 show that it enough just check the vertices of Ω for optimal solutions. Unfortunately, there may be a lot of these, even for low-dimensional problems. In fact, there are as many corners as there are subsets of r linearly independent columns in A. Thus, the number of basic points is somewhere between 1 (since the Fundamental Theorem proves their existence) and $\binom{n}{r}$,– but, at least, *the number is finite*!

The objective function may have the same optimal value at several vertices. Also, if x_a^* and x_b^* are optimal solutions, and hence $f(x_a^*) = f(x_b^*)$, then even $\theta x_a^* + (1-\theta) x_b^*$, $0 \leq \theta \leq 1$, is optimal (recall what we know about the set of optimal solutions!).

5 The Simplex Algorithm

The name *simplex* refers to the shape of Ω, see the literature. The *Simplex Algorithm* is reported to have been discovered by G. B. Dantzig in 1947 and published around 1949. The idea of the Simplex Algorithm is to search for the minimum by going from vertex to vertex (from basic point to basic point) in Ω.

As G. Strang says in his book about Linear Algebra, the algorithm in the pre-computer days, with its so-called numerical *tableau*-s, was surrounded with an "aura of mystery". The *tableau* was just a practical way of setting up the matrix calculus, but the days of the *tableau* are gone forever.

Unless you are interesting in the history of computation, there is no point in doing a simplex solution with pen and paper anymore, but the geometric idea is quite elegant, and coding the algorithm is challenging. Below we discuss the basic steps in the iteration and the question of finding a starting point.

5.1 The Simplex Iterative Step

We assume that the problem has the standard form and that we are located in a basic point,

$$x = \left[\begin{array}{c} x_B \\ 0 \end{array} \right], \tag{34}$$

where the partition is according to $A = [B\ N]$, where B non-singular, and such that

$$Ax = [B\ N] \left[\begin{array}{c} x_B \\ 0 \end{array} \right] = Bx_B = b. \tag{35}$$

More generally, with perhaps a little confusing notation, we split a general x in the same way,

$$A \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] = Bx_1 + Nx_2 = b. \tag{36}$$

Hence,

$$x_1 = B^{-1} (b - Nx_2) = x_B - B^{-1} N x_2. \tag{37}$$

We stay at a point where $x_1 = x_B$ and $x_2 = 0$, and we now try to change one of the components $(x_2)_j$ of x_2 (make it a little positive) such that the objective function,

$$f(x) = c'x = c_1' x_1 + c_2' x_2 \tag{38}$$

decreases.

Task 16: *Prove that*

$$f(x) = c'x = c_1' x_B + \left(c_2' - c_1' B^{-1} N \right) x_2 \tag{39}$$

and that we already have the solution of the full problem if all components of $\left(c_2' - c_1' B^{-1} N \right)$ *are non-negative!*

OK, assume that $\left(c_2' - c_1' B^{-1} N \right)_j$ is negative, such that $f(x)$ really decreases if the corresponding $(x_2)_j$ increases from 0. But this affects x_1, which will now start to change from x_B.

Task 17: *Assume that all components of* x_1 *increase when* $(x_2)_j$ *increases. Show that*

$$\min_{x \in \Omega} c'x = -\infty. \tag{40}$$

In general, some components of x_1 will increase and some will decrease, and this situation is illustrated in Fig. 2.

Assume that the basic point was non-degenerate (that is, all components of x_B different from 0). We now increase $(x_2)_j$ until some of the components in x_1 hit 0. We then have a new feasible point

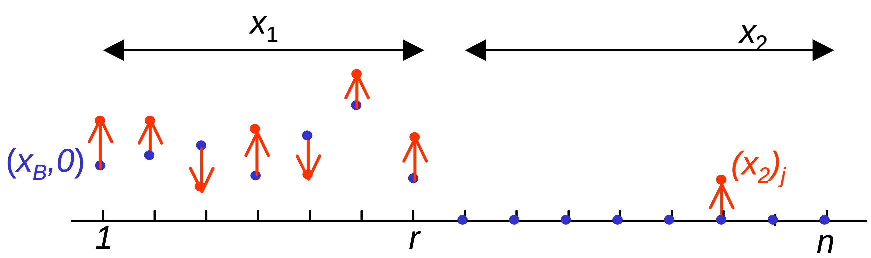

Figure 2: Before we start to move $(x_2)_j$, we have the basis solution (blue). When $(x_2)_j$ changes, this affects x_1, where some components increase and some decrease.

with at most r components different from 0, and where $f(x)$ has decreased. If the corresponding columns in A are linearly independent, we have a new basic point.

Task 18: *Prove that the Simplex algorithm always converges if all basic points are non-degenerate and the objective is bounded below on Ω.*

What happens if one component, $(x_B)_k$, in x_B is 0? When we increase $(x_2)_j$, the component may increase (good!), or decrease (prohibited!). If we had bad luck, we replace the column corresponding to $(x_B)_k$ by another (linearly independent) column from the set corresponding to x_2 and try again, and again, and again, $\cdots$.

There are a lot of smart tricks to make this choice, which we will not go into here.

This is basically it for the Simplex Algorithm, as soon as we have been able to start (see below).

Some additional notes:

- It is straightforward to construct a generalized Simplex Algorithm for bounds of the form
$$l_i \le x_i \le u_i, \ i = 1, \cdots, n. \tag{41}$$
This will save us from increasing the number of unknowns.
- If we LU-factorize B once, we can update the factorization with the new column without making a complete new factorization (N&W, Sec. 13.5).
- It is often preferable to take the "steepest ridge" (fastest decrease in the objective) out from where we are (N&W, Sec. 13.5).

5.2 Starting the Simplex Method

In general, the Simplex method consists of two phases:

- Phase 1: *Find a first basic point*
- Phase 2: *Solve the problem*

The following outlines one possible way for the phase 1 algorithm:

- Change the signs in $Ax = b$ so that $b \ge 0$.

- Introduce additional variables $y \in \mathbb{R}^r$ and solve the extended problem

$$\begin{aligned} &\min\left(y_1 + \cdots + y_r\right), \\ &[A \;\; I]\begin{bmatrix} x \\ y \end{bmatrix} = b, \\ &x, y \geq 0. \end{aligned} \tag{42}$$

(Note that $\begin{bmatrix} 0 \\ b \end{bmatrix}$ already *is* a basic point for this problem!).

Assume that the solution is

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \tag{43}$$

Task 19: *Prove that if* $y_0 \neq 0$*, then the original problem is infeasible* ($\Omega = \varnothing$).

Task 20: *Prove that if* $y_0 = 0$*, then* x_0 *is a basic point* (= *possible start point for the original problem*).

6 Final Notes

The following notes reflect my own current knowledge, and are not necessarily up-to-date.

One famous problem in LP has been whether there are algorithms for $x \in \mathbb{R}^n$ where the number of operations grows as n^α for some positive α.

The Simplex algorithm has long been demonstrated to require an exponential number of operations in the worst case: It possible to show, for a certain hypercube-shaped Ω with 2^n vertices, that the algorithm may be set up so that it is forced to visit all vertices (*Klee–Minty–Cheval counterexample).* The selection of the columns to include is essential, and the "steepest ridge" is assumed for the counterexample However, such bad performance has never been observed in practical problems.

A breakthrough came in the late 70-ties when *Khachiyan* (1978) published a so-called *interior point method* which encloses the solution in a sequence of ellipsoids containing x^* with L significant digits after $\mathcal{O}\left(n^4 L\right)$ operations.

The *Karmankar* method from 1984 is simpler and slightly faster, $\mathcal{O}\left(n^{3.5} L\right)$.

As far as I know, the current record is, for so-called *Interior Barrier Primal–Dual* methods, $\mathcal{O}\left(n^3 L\right)$.

It should be also be mentioned that in spite of these theoretical bounds, the Simplex algorithm is still compatible to interior point methods for medium sized problems. Interior Point Methods are discussed in N&W in Chapter 14.

It still appears to be open whether there exists an algorithm such that

$$\#\text{ operations} \propto n^p r^q \;,\; p, q > 0. \tag{44}$$

It has also been discovered that interior point algorithms have been proposed earlier without getting the notice they deserved. One such approach, discussed in G. Strang's book on Linear

Algebra, is *Dikin's Method* from 1967. This is a trust region method of the form

$$
\begin{aligned}
x_{k+1} &= \arg\min c'x, \\
Ax &= b, \\
x &\in \mathcal{E}\left(x_k\right),
\end{aligned}
\tag{45}
$$

where $\mathcal{E}\left(x_k\right)$ is the ellipsoid

$$
E\left(x_k\right) = \left\{ x \; ; \; \sum_{i=1}^{n} \left(\frac{x_i - \left(x_k\right)_i}{\left(x_k\right)_i} \right)^2 \leq 1 \right\}. \tag{46}
$$

Task 21: *Prove that the solution of Dikin's Problem in Eq. 45 is*

$$
\begin{aligned}
x_{k+1} &= x_k - \frac{DP_M Dc}{\|P_M Dc\|}, \\
D &= \operatorname{diag}\left(x_k\right), \\
P_M &= \text{Projection operator on } \mathcal{N}\left(AD\right).
\end{aligned}
\tag{47}
$$

A barrier method related to the $\mathcal{O}\left(n^3 L\right)$-algorithm was also proposed already in the 1950-ies by the Norwegian Nobel laureate in Economy, Ragnar Frisch,

$$
\begin{aligned}
x^*\left(\mu\right) &= \min\left(c'x + \mu p\left(x\right)\right), \\
Ax &= b \\
p\left(x\right) &= -\sum_{i=1}^{n} \ln x_i \\
\mu &\rightarrow 0,
\end{aligned}
\tag{48}
$$

(R. Frisch: *The Logarithmic Potential Method for Convex Programming*, Report University of Oslo, 1955).

We shall return to this method after discussing penalty and barrier methods.