THE LP PROBLEM IN STANDARD FORM

$$\min_{x \in \mathbb{R}^n} c'x,$$
$$Ax = b, \; x \geq 0.$$

- $x \geq 0$ means $x_i \geq 0, \; i = 1, \cdots, n$.

- A of size $r \times n$ is supposed to have *full rank* r.

- Ω is a **polytope (polyhedron** if bounded**).**

- This is a *convex* optimization problem $\Rightarrow$ KKT conditions sufficient for a global minimum.

GEOMETRY OF THE FEASIBLE SET

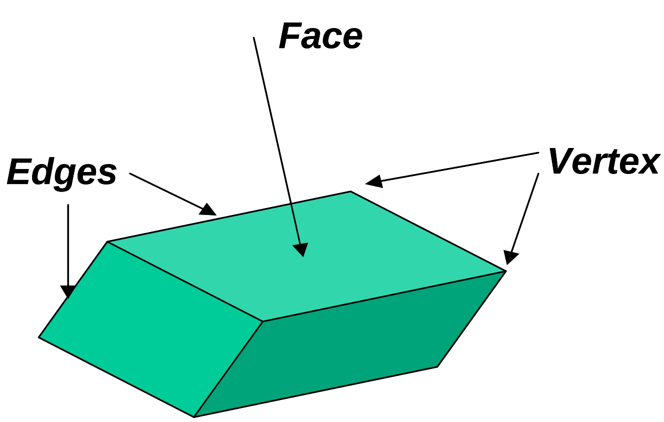
Definition: The point $x_e \ \in \partial\ \Omega$ (= the boundary of Ω) is an *extreme point* if

$$x_e = \theta y + (1-\theta)\, z \ , \ \ y, z \in \Omega \ , \ \ 0 < \theta < 1$$

implies that $y = z = x_e$.

Where are the extreme points for a *line segment*, for $\mathbb{R}$ and $\mathbb{R}^n_+$, a *cube*, and a *sphere* (all sets closed)?

The extreme points for Ω are the *vertices*.



Definition: A feasible point x ($x \geq 0$, $Ax = b$) is called a *basic point* if there is an index set $\mathcal{B} = \{i_1, \cdots, i_r\}$, where the corresponding subset of columns of A,

$$\left\{a_{i_1}, \cdots, a_{i_r}\right\},$$

are linearly independent, and $x_i = 0$ for all $i \notin \mathcal{B}$.

If x_i happens to be 0 also for some $i \in \mathcal{B}$, we say that the basic point is *degenerate.*

For a basic point, the corresponding $r \times r$ matrix

$$B = \left[a_{i_1}, \cdots, a_{i_r}\right],$$

will be *non-singular*, and the equation $Bx_B = b$ has a unique solution.

The Fundamental Theorem for LP (N&W Theorem 13.2):

1. *If* $\Omega \neq \emptyset$, *it contains basic points.*

2. *If there are optimal solutions, there are optimal basic points (basic solutions).*

Theorem (N&W Theorem 13.3): *The basic points are the extreme points of* Ω.

The number of basic points is between 1 (because of the first statement in the Fundamental Theorem) and $\binom{n}{r}$.

THE SIMPLEX ALGORITHM

- The *Simplex Algorithm* is reported to have been discovered by G. B. Dantzig in 1947.

- The idea of the Simplex Algorithm is to search for the minimum by going from vertex to vertex (from basic point to basic point) in Ω.

- Hand calculations are *never used* anymore!

The Simplex Iteration Step

We assume that the problem has the standard form, and that we are located in a basic point which, after a rearrangement of variables, has the form

$$x = \begin{bmatrix} x_B \\ 0 \end{bmatrix}.$$

The partition is therefore according to $A = [B\ N]$, where B is non-singular, and

$$Ax = [B\ N] \begin{bmatrix} x_B \\ 0 \end{bmatrix} = Bx_B = b.$$

Split a general $x \in \Omega$ in the same way,

$$Ax = [B\ N] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Bx_1 + Nx_2 = b.$$

Hence,

$$x_1 = B^{-1}(b - Nx_2) = x_B - B^{-1}Nx_2.$$

Note also that

$$\begin{aligned} f(x) &= c'x = [c_1\ c_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= c_1'x_1 + c_2'x_2 \\ &= c_1'\left(x_B - B^{-1}Nx_2\right) + c_2'x_2 \\ &= c_1'x_B + \left(c_2' - c_1'B^{-1}N\right)x_2 \end{aligned}$$

Around $[x_B\ 0]'$, we may express both x_1 and $f(x)$ in terms of x_2.

We are located at $x_1 = x_B$, $x_2 = 0$, and try to change one of the components $(x_2)_j$ of x_2 so that

$$f(x) = c_1' x_B + \left(c_2' - c_1' B^{-1} N \right) x_2$$

decreases.

- If $\left(c_2' - c_1' B^{-1} N \right) \geq 0 \Rightarrow$ **FINISHED!**

Assume that $\left(c_2' - c_1' B^{-1} N \right)_j < 0$:

- If all components of x_1 increase when $(x_2)_j$ increases, then

$$\min c' x = -\infty.$$

$\Rightarrow$ **FINISHED!**

If not, we have the situation shown in Fig. 1.

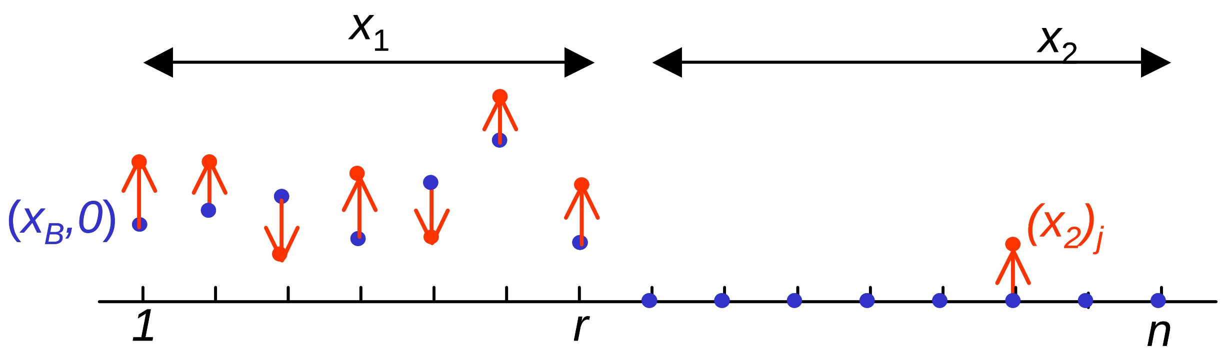


Figure 1: Change in x_1 when $(x_2)_j$ increases from 0.

- The Simplex algorithm always converges if all basic points are non-degenerate.

- Degenerate basic point: *Try a different component of* x_2. (**FINISHED** if impossible!)

- It is straightforward to construct a generalized Simplex Algorithm for bounds of the form

$$l_i \leq x_i \leq u_i, \ i = 1, \cdots, n.$$

- If we LU-factorize B once, we can update the factorization with the new column without making a complete new factorization (N&W, Sec. 13.4).

- It is often preferable to take the "steepest ridge" (fastest decrease in the objective) out from where we are (N&W, Sec. 13.5).

Starting the Simplex Method

The Simplex method consists of two phases:

- Phase 1: *Find a first basic point*
- Phase 2: *Solve the original problem*

The Phase 1 algorithm:

1. Turn the signs in $Ax = b$ so that $b \geq 0$.
2. Introduce additional variables $y \in \mathbb{R}^r$ and solve the extended problem

$$\min\left(y_1 + \cdots + y_r\right),$$
$$[A \;\; I]\begin{bmatrix} x \\ y \end{bmatrix} = b, \quad x, y \geq 0.$$

   (Note that $[0 \; b]'$ already is a basic point for the extended problem!).

Assume that the solution of the extended problem is

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

- If $y_0 \neq 0$, then the original problem is infeasible ($\Omega = \varnothing$).

- If $y_0 = 0$, then x_0 is a basic point ($=$ possible start for the original problem).

- This is not the only Phase 1 algorithm.

1 EPILOGUE

- Open Problem: *Are there LP algorithms of polynomial complexity*?

- The Simplex Method has exponential complexity in the worst case (*Klee–Minty–Cheval counterexample*)

- Interior Point Methods (Khachiyan, 1978): $\#Op \propto \mathcal{O}\left(n^4 L\right)$

- Karmankar (1984): $\#Op \propto \mathcal{O}\left(n^{3.5} L\right)$

- Current record (?): *Interior Barrier Primal–Dual methods*, $\#Op \propto \mathcal{O}\left(n^3 L\right)$. (We return to this method after discussing penalty and barrier methods)

- Solving large LP problems is BIG business!

- Entering data into the computer for large LP problems is a lot of work. Look up a description of the industry standard "*MPS Data Format*" on the *internet.*