Preface

The following problem deals with optimal fuel consumption for a ship travelling a prescribed distance in a prescribed time. However, it must be admitted that the example is not particularly realistic.

Ship resistance (*drag*) in the sea is a quite complicated topic where there are different contributions to the total drag (See Wikipedia, `http://en.wikipedia.org/wiki/Drag_(physics)`). Apart from plain friction drag acting on the submerged part of the ship, there is drag from pushing water away, and the very important drag caused by the wake after the ship (The waves in the wake carry energy away from the ship).

For small, blunt ships (e.g. fishing boats and ferries), the friction force is about quadratic in the velocity. If we therefore assume that the force F is equal to Av^2, the necessary power to keep the ship's speed equal to v will be

$$P = \frac{F \times \Delta S}{\Delta t} = \frac{F \times (v\Delta t)}{\Delta t} = Av^3,$$

and therefore *cubic* in the velocity. Other expressions for the drag are of the form

$$F = A\dot{v} + Bv^2$$

Therefore, the final expression for the necessary power in the problem,

$$P \propto (\alpha\dot{v} + v)^2 \, ,$$

and using $\alpha = 1$ (after non-dimensionalization) is therefore chosen for convenience and not for realism.

1 Problem 3.39

Force:

$$F \propto \dot{v} + v$$

Fuel consumption

$$\propto F^2$$

Consider minimizing the fuel consumption

$$J(v) = \int_0^T (\dot{v} + v)^2 \, dt$$

when

$$\begin{aligned} v(0) &= 0, \\ v(T) &= \text{free} \end{aligned}$$

and

$$\int_0^T v dt = 1.$$

Lagangian:

$$L(v) = \int_0^T \left[(\dot{v} + v)^2 + \lambda v \right] dt$$

Euler:

$$\begin{aligned} \frac{d}{dt}(2(\dot{v} + v)) - 2(\dot{v} + v) - \lambda &= 0, \\ \ddot{v} + \dot{v} - \dot{v} - v &= \frac{\lambda}{2}, \\ \ddot{v} - v &= \frac{\lambda}{2} \end{aligned}$$

General solution:

$$v(t) = Ae^t + Be^{-t} - \frac{\lambda}{2}.$$

Boundary condition at 0:

$$v(0) = A + B - \frac{\lambda}{2} = 0,$$

Natural condition at $t = T$:

$$\begin{aligned} \frac{\partial}{\partial \dot{v}} \left((\dot{v} + v)^2 + \lambda v \right)(T) &= 0, \\ 2(\dot{v}(T) + v(T)) &= 0, \end{aligned}$$

This gives:

$$Ae^T + Be^{-T} - \frac{\lambda}{2} + Ae^T - Be^{-T} = 0$$

or

$$2Ae^T - \frac{\lambda}{2} = 0,$$

We then find the values of A and B in tems of T and λ:

$$\begin{aligned} A &= \frac{\lambda}{4}e^{-T}, \\ B &= -\frac{\lambda}{4}e^{-T} + \frac{\lambda}{2} \end{aligned}$$

The constraint:

$$\begin{aligned} \int_0^T \left(Ae^t + Be^{-t} - \frac{\lambda}{2} \right) dt &= \\ A\left(e^T - 1\right) + B\left(1 - e^{-T}\right) - \frac{\lambda}{2}T &= 1. \end{aligned}$$

Solution:

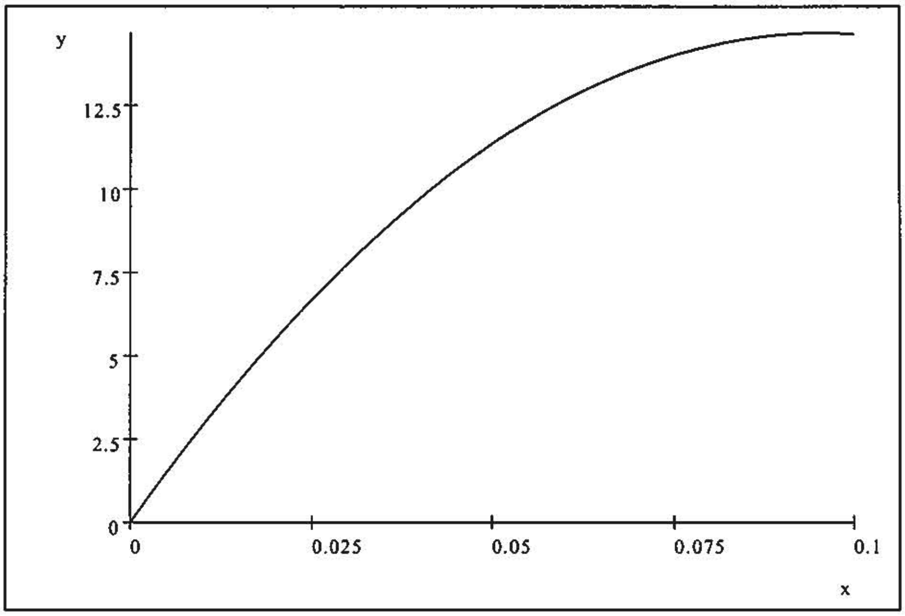
$$\lambda = \frac{1}{-\frac{1}{2}T + \frac{1}{4}\left(1 - e^{-T}\right) + \left(-e^{-T} + 1\right)\left(-\frac{1}{4}e^{-T} + \frac{1}{2}\right)}$$

Thus,

$$
\begin{aligned}
A &= \frac{\lambda}{4}e^{-T}, \\
B &= -\frac{\lambda}{4}e^{-T} + \frac{\lambda}{2} \\
\lambda &= \frac{1}{-\frac{1}{2}T + \frac{1}{4}e^{-T}\left(e^{T} - 1\right) + \left(-e^{-T} + 1\right)\left(-\frac{1}{4}e^{-T} + \frac{1}{2}\right.}
\end{aligned}
$$

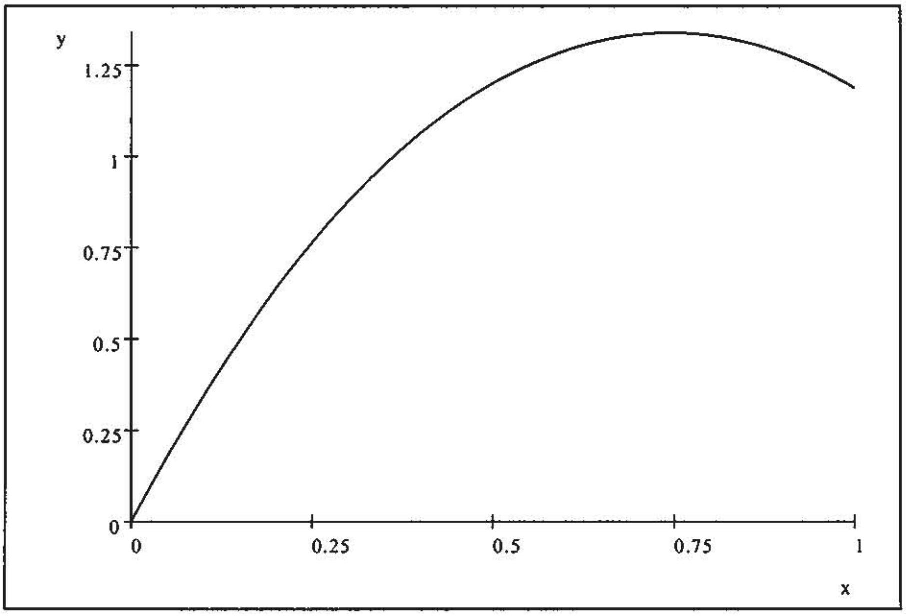
$$T = 0.1$$

$$Ae^{t} + Be^{-t} - \frac{\lambda}{2}$$



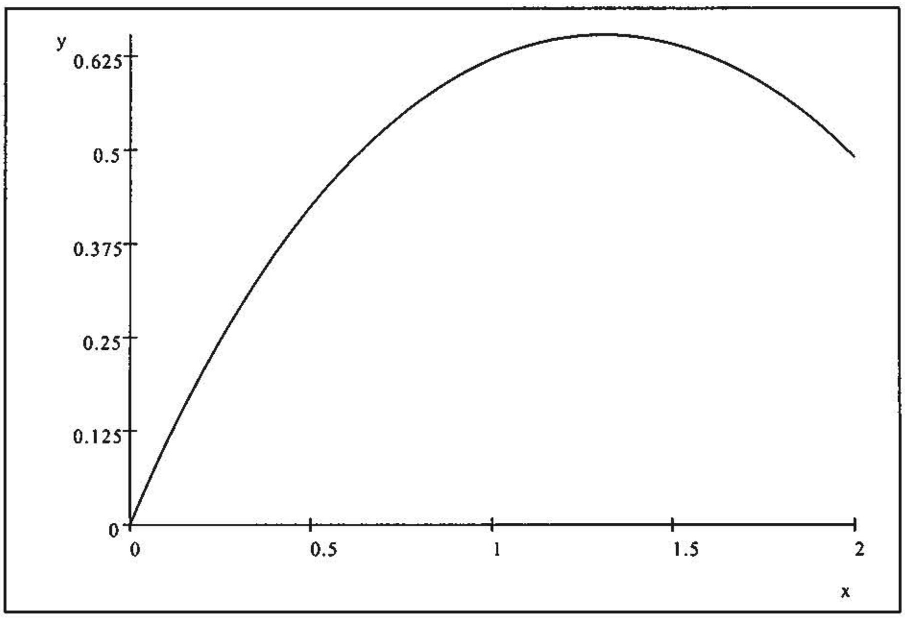
$$T = 1$$

$$Ae^{t} + Be^{-t} - \frac{\lambda}{2}$$



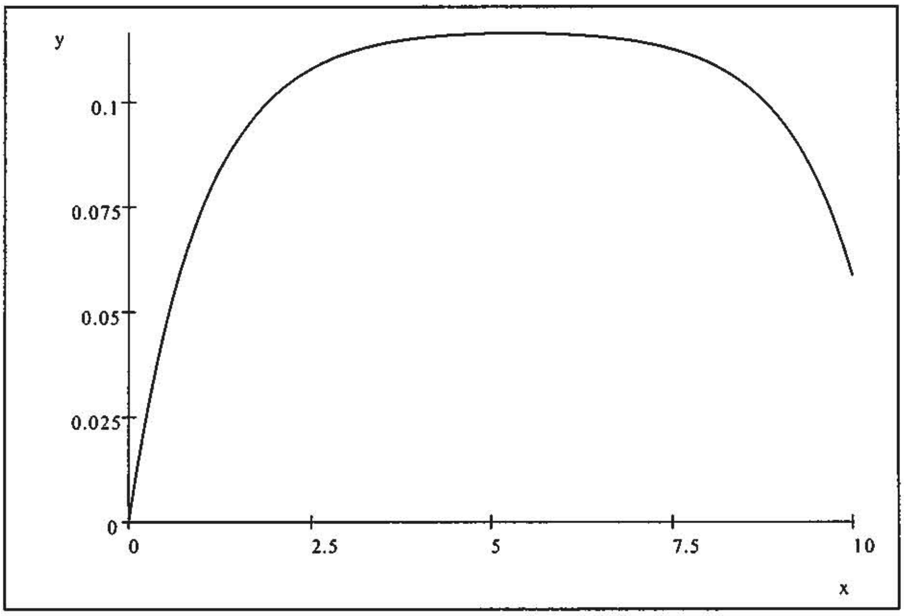
$$T = 2$$

$$Ae^{t} + Be^{-t} - \frac{\lambda}{2}$$



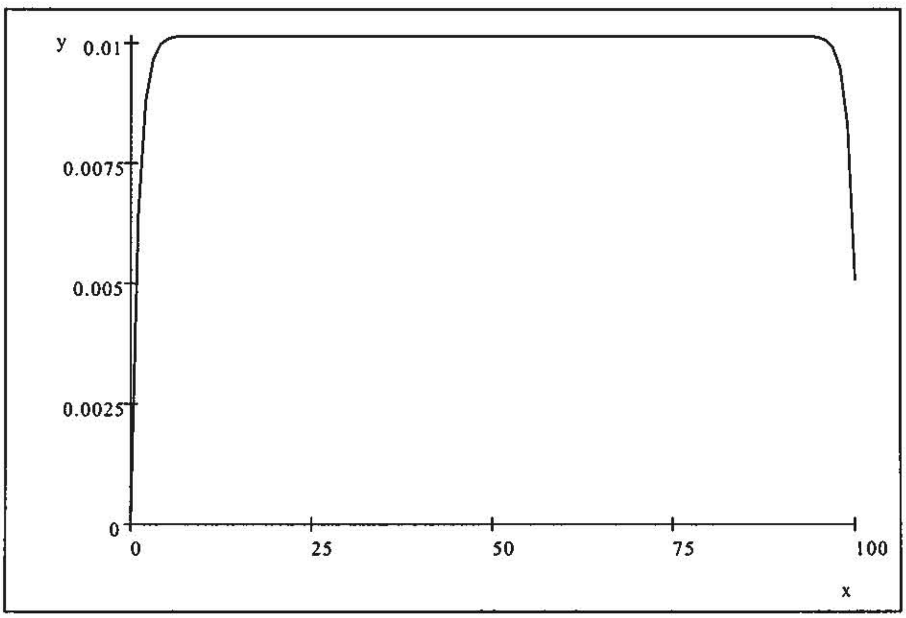
$$T = 10$$

$$Ae^{t} + Be^{-t} - \frac{\lambda}{2}$$



$$T = 100$$

$$Ae^{t} + Be^{-t} - \frac{\lambda}{2}$$



Is the solution unique??