VARIATIONAL CALCULUS

PARTIAL DIFFERENTIAL EQUATIONS AND THE FINITE ELEMENT METHOD

TMA 4180 Optimization Theory

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THE CORRESPONDENCE

Variational functional $F(u)=\int_D f\left(x,u,u_x,u_y,\cdots\right)d^n x,\ \mathbf{x}\in \mathrm{R}^n$

$\Updownarrow$

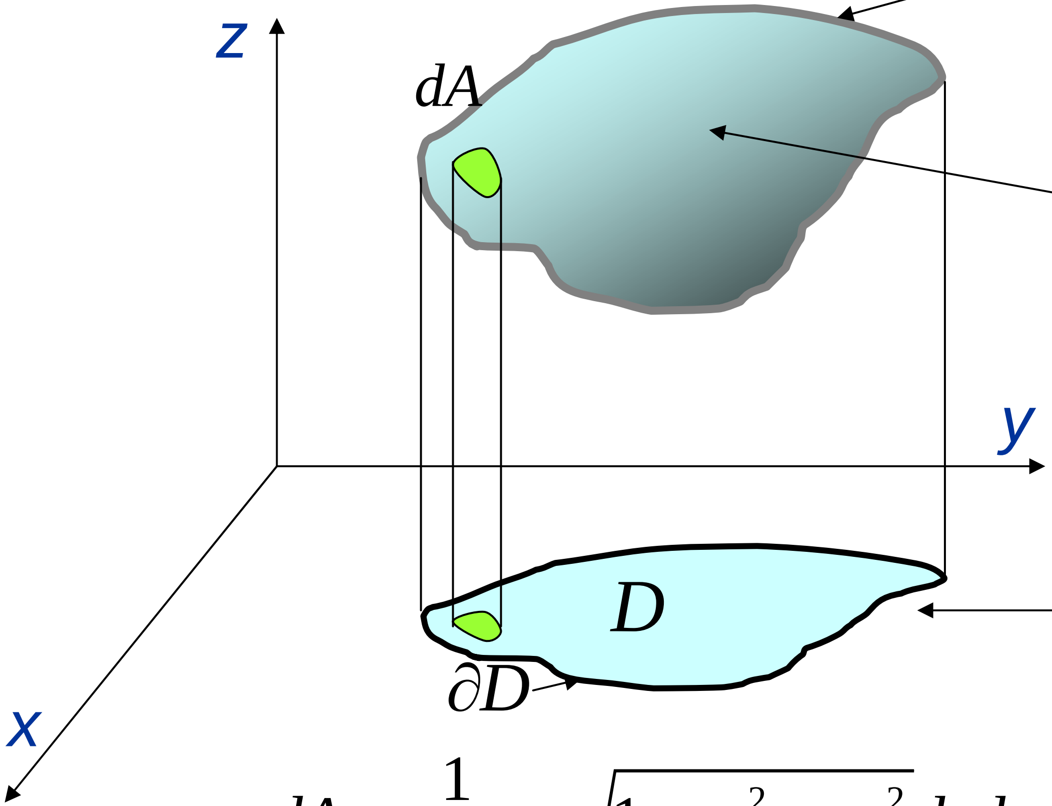
Euler equation

$||$

Partial differential equation $P\left(x,u,u_x,u_y,\cdots\right)=0$

Many equations in physics are Euler equations of some variational functional

THE MINIMAL AREA (SOAP-FILM) EQUATION
(Troutman, p. 74 – 76)



Wireframe:

$$z=\gamma(x,y),(x,y)\in\partial D$$

Soap-film:

$$z=u(x,y)$$

Projection

$$dA=\frac{1}{\mathbf{n}\bullet\mathbf{k}}=\sqrt{1+u_x^{\;2}+u_y^{\;2}}\,dxdy,\quad u_x=\frac{\partial u}{\partial x},u_y=\frac{\partial u}{\partial y}$$

$$\mathcal{D}=\left\{u\in C^1(D);u(x,y)=\gamma(x,y),(x,y)\in\partial D\right\}$$

$$\min_{u \in \mathcal{D}} S(u)$$

$$S(u) = \iint dA = \iint_D \sqrt{1 + {u_x}^2 + {u_y}^2}\, dxdy$$

$$\mathcal{D} = \left\{ u \in C^1(D); u(x,y) = \gamma(x,y), (x,y) \in \partial D \right\}$$

The Gâteaux derivative:

$$\delta S(u;v) = \left. \frac{dS(u + \varepsilon v)}{d\varepsilon} \right|_{\varepsilon=0} = \iint_D \frac{u_x v_x + u_y v_y}{\sqrt{1 + {u_x}^2 + {u_y}^2}} dxdy$$

Problem 1: Prove that S is *strictly convex*:

$$S(u+v) - S(u) \geq \delta S(u;v)$$

$$"=" \text{ only when } v = 0$$

Hint: *The function*

$$f(y,z) = \sqrt{1+y^2+z^2}$$

is strictly convex.

Derivation of the Euler Equation
(simplified derivation when $|u_x|, |u_y| << 1$):

$$S(u) = \iint_D \sqrt{1 + u_x{}^2 + u_y{}^2}\, dxdy$$

$$\approx \iint_D \left(1 + \frac{1}{2}\left(u_x{}^2 + u_y{}^2\right)\right) dxdy = |D| + \frac{1}{2}\iint_D \left(u_x{}^2 + u_y{}^2\right) dxdy$$

The ***Dirichlet Functional***:

$$P(u) = \frac{1}{2}\iint_D \left(u_x{}^2 + u_y{}^2\right) dxdy$$

$$\delta P(u;v) = \iint_D \left(u_x v_x + u_y v_y\right) dxdy$$

Let $\mathbf{U} = (U, W) = \nabla u = (u_x, u_y)$:

$$\delta P(u;v) = \frac{1}{2}\iint_D (Uv_x + Wv_y)\,dxdy$$

$$\overset{*)}{=} \frac{1}{2}\iint_D \left[(Uv)_x + (Wv)_y - U_x v - W_y v\right] dxdy$$

$$= \frac{1}{2}\iint_D \nabla\bullet(\mathbf{U}v)\,dxdy - \frac{1}{2}\iint_D (\nabla\bullet\mathbf{U})\,v\,dxdy$$

$$= \frac{1}{2}\iint_{\partial D} \mathbf{n}\bullet(\mathbf{U}v)\,ds - \frac{1}{2}\iint_D (\nabla\bullet\mathbf{U})\,v\,dxdy$$

$$= 0 - \frac{1}{2}\iint_D (u_{xx} + u_{yy})\,v\,dxdy$$

$v(x, y) = 0$ for $(x, y) \in \partial D$!

*) $\nabla\bullet(\mathbf{U}v) = (\nabla\bullet\mathbf{U})v + \mathbf{U}\bullet\nabla v$

Conclusion: $\delta P(u;v)=0$ if

$$u_{xx}+u_{yy}=\nabla^2 u=0,$$
$$u(x,y)=\gamma(x,y), \quad (x,y)\in\partial D.$$

The **Dirichlet problem** for the Laplace equation

Problem 2: Show that the Euler equation for the *complete* area functional S is:

$$\left(1+u_y^2\right)u_{xx}-2u_x u_y u_{xy}+\left(1+u_x^2\right)u_{yy}=0$$

(Called the ***Plateau Equation***)

THE FINITE ELEMENT METHOD – AN EXAMPLE

Dirichlet problem for the Laplace equation
(stationary diffusion, temperature distributions, etc.):

$$u_{xx} + u_{yy} = \nabla^2 u = 0,$$
$$u(x,y) = \gamma(x,y), \quad (x,y) \in \partial D.$$

We use the expression for the derivative of the
Dirichlet functional:

$$\delta P(u;v) = \frac{1}{2} \iint_D \left(u_x v_x + u_y v_y\right) dxdy = \frac{1}{2} \iint_D \nabla u \bullet \nabla v \, dxdy = 0$$

Basic idea:

1. Expand u in a set of *basis functions* $\left\{ \varphi_i \right\}_{i=1}^{N}$:

$$u = u_0 + \sum_{i=1}^{N} \alpha_i \varphi_i ,$$

$$u_0\left(x, y\right) = \gamma\left(x, y\right), \left(x, y\right) \in \partial D$$

Some function satisfying the boundary cond.

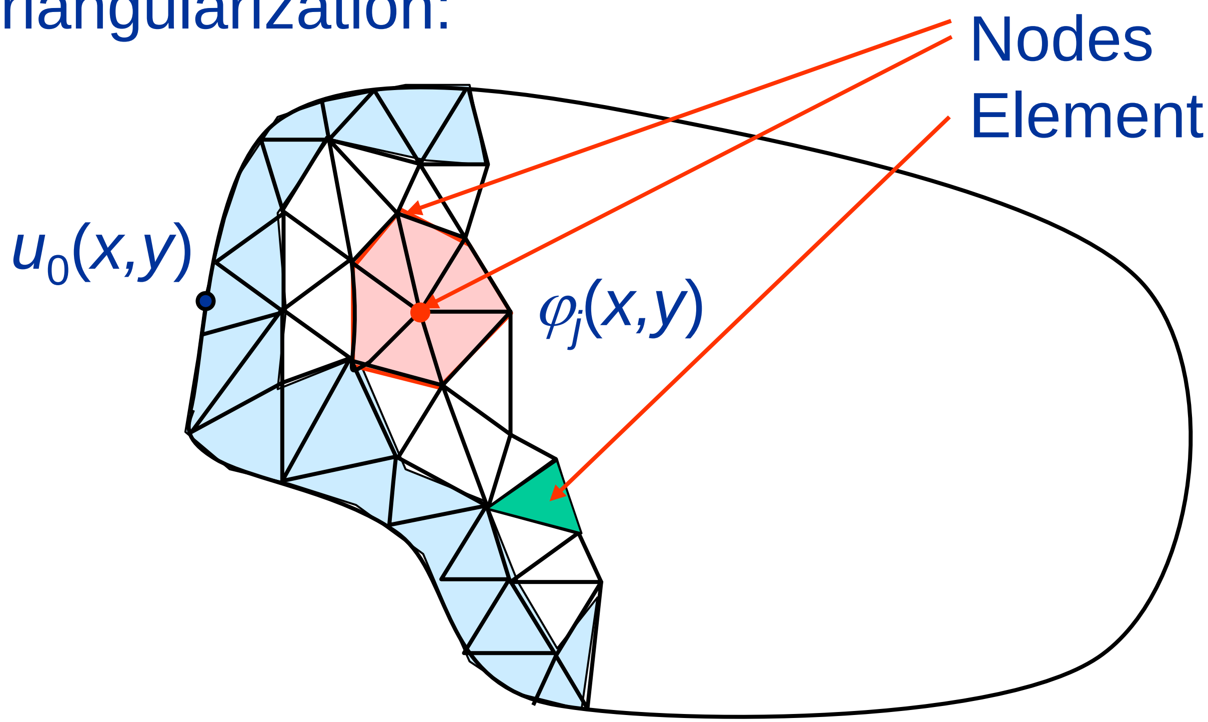
2. Require that

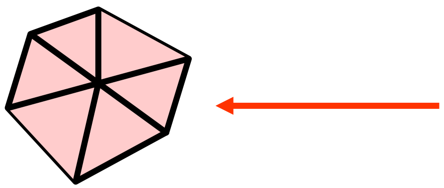
$$\delta P\left(u; v\right) = 0 \quad \forall v \in \left\{ \varphi_i \right\}_{i=1}^{N}$$

3. Insert and solve for $\boldsymbol{\alpha} = \left\{ \alpha_i \right\}_{i=1}^{N}$

$$
\begin{aligned}
&2\delta P\left(u,\varphi_j\right)\\
&=\iint_D \left(u_x\varphi_{jx}+u_y\varphi_{jy}\right)dxdy\\
&=\iint_D \left[\left(u_{0x}+\sum_{i=1}^{N}\alpha_i\varphi_{ix}\right)\varphi_{jx}+\left(u_{0y}+\sum_{i=1}^{N}\alpha_i\varphi_{iy}\right)\varphi_{jy}\right]dxdy\\
&=\iint_D \left[u_{0x}\varphi_{jx}+u_{0y}\varphi_{jy}\right]dxdy+\sum_{i=1}^{N}\left\{\iint_D \left(\varphi_{jx}\varphi_{ix}+\varphi_{jy}\varphi_{iy}\right)dxdy\right\}\alpha_i\\
&=b_j+\sum_{i=1}^{N}a_{ji}\alpha_i\\
&=\left(\mathbf{b}-\mathbf{A}\boldsymbol{\alpha}\right)_j=0
\end{aligned}
$$

Triangularization:





$$\varphi_j(x,y) = \begin{cases} 1 \text{ at the center node} \\ 0 \text{ at the outer boundaries} \end{cases}$$

$$\iint_D \left(\varphi_{jx}\varphi_{ix} + \varphi_{jy}\varphi_{iy} \right) dxdy \neq 0$$ only for neighboring nodes!

We need to solve the linear system

$$\mathbf{A\alpha} = \mathbf{b}$$

$$b_i = -\iint_D \left[u_{0x}\varphi_{jx} + u_{0y}\varphi_{jy} \right] dxdy$$

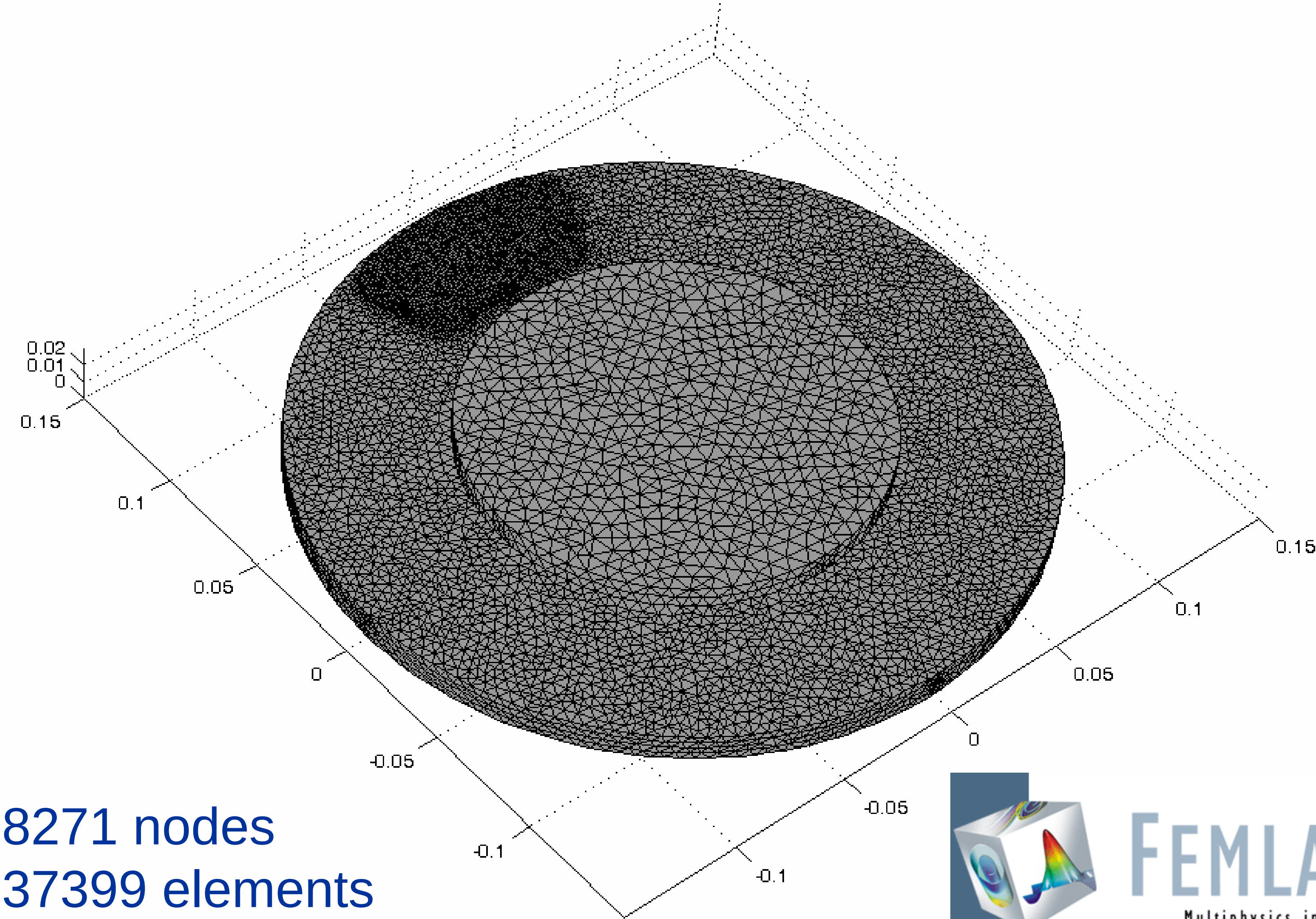
$$a_{ij} = \iint_D \left[\varphi_{ix}\varphi_{jx} + \varphi_{iy}\varphi_{jy} \right] dxdy$$

- The A-matrix is *positive definite* and *sparse*

- Such a solution is called a ***weak solution*** of the differential equation

(We are able to use ”tent-functions” for φ, and then u_{xx} and u_{yy} do not exist).

MODELLING OF HEAT IN A DISK-BRAKE



28271 nodes
137399 elements

FEMLAB
Multiphysics in MATLAB

$$\rho C\frac{\partial T}{\partial t} - \nabla \cdot (k\nabla T) = \rho C\omega\left(y\frac{\partial T}{\partial x} - x\frac{\partial T}{\partial y}\right)$$