TMA 4180 Optimeringsteori

Spring 2012

Troutman, Exercise 3.29

Exercise 3.29 in Troutman, p. 92, is an example where Theorem 3.16 about convex, constrained problems fails to apply. As stated in T., the problem consists of solving

$$\min \int_0^\pi y'^2(x)\,dx, \tag{1}$$

when $y(0) = y(\pi) = 0$, and

$$G(y) = \int_0^\pi y^2(x)\,dx = 1. \tag{2}$$

Following the standard procedure, we form the Lagrangian,

$$L(y,\lambda) = \int_0^\pi \left(y'(x)^2 + \lambda y(x)^2\right) dx, \tag{3}$$

from which it follows that L is strictly convex for $\lambda \geq 0$.

The Euler equation becomes

$$\frac{d}{dx}\left(2y'(x)\right) - 2\lambda y(x), \tag{4}$$

leading to

$$y''(x) - \lambda y(x) = 0, \tag{5}$$

with the well-known general solution

$$Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}. \tag{6}$$

For $\lambda \geq 0$, the only solution satisfying $y(0) = y(\pi) = 0$ is $y_0(x) = 0$. In other words, for $\lambda \geq 0$, *no solution exists from which it is possible to satisfy the constraint*!

For $\lambda < 0$ we may suspect that the functional is no longer convex, since

$$f(x,y,z) + \lambda g(y) = z^2 + \lambda y^2 \tag{7}$$

is not. Let us anyway write, as we do for Fourier series,

$$\lambda = -\mu^2, \tag{8}$$

so that the Euler equation and boundary conditions become

$$\begin{gathered} y''(x) + \mu^2 y(x) = 0, \\ y(0) = y(\pi) = 0. \end{gathered} \tag{9}$$

It is easy to see that nontrivial solutions only occur for $\mu = n$, $n = 1, 2, \cdots$, and become

$$y_n = \sqrt{\frac{2}{\pi}} \sin nx, \;\; n = 1, 2\cdots. \tag{10}$$

Thus, $\{y_n\}_{n=1,2\cdots}$ become stationary points for L, $\delta L(y_n; v) = 0$, and are therefore candidates for minima.

In order to come further it appears to be necessary to apply some theory from Fourier analysis, namely *Parseval's Equality* (Kreyszig). In the present case, $\{y_n\}_{n=1,2\cdots}$ is a well-known complete orthogonal basis on $L^2[0,\pi]$. Leaving out some details about convergence etc., we may write

$$y(x) = \sum_{n=1}^{\infty} b_n \sin nx, \tag{11}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} y(x) \sin nx. \tag{12}$$

By Parseval's Equality,

$$\int_0^{\pi} y(x)^2 \, dx = \frac{\pi}{2} \sum_{n=1}^{\infty} b_n^2. \tag{13}$$

Similarly, since

$$y'(x) = \sum_{n=1}^{\infty} n b_n \cos nx, \tag{14}$$

we obtain

$$\int_0^{\pi} y'(x)^2 \, dx = \frac{\pi}{2} \sum_{n=1}^{\infty} n^2 b_n^2. \tag{15}$$

The problem has then been included in the algebraic optimization problem

$$\min_{\{b_n\}} \sum_{n=1}^{\infty} n^2 b_n^2 \tag{16}$$

when

$$\sum_{n=1}^{\infty} b_n^2 = \frac{2}{\pi}.$$

The solution is obvious,

$$\begin{aligned} b_1 &= \sqrt{\frac{2}{\pi}}, \\ b_n &= 0, \;\; n = 2, 3 \cdots , \end{aligned} \tag{17}$$

which corresponds to

$$\sqrt{\frac{2}{\pi}} \sin x, \tag{18}$$

and the first stationary point for the Lagrangian.

This *is* the unique solution, even if $y'^2 - y^2$ is definitely not convex. Let us therefore check the convexity of L more carefully for

$$\lambda = -\mu^2. \tag{19}$$

Now

$$L(y+v) - L(y) = \int_0^{\pi} 2\left(y'v' - \mu^2 yv\right) dx + \int_0^{\pi} \left(v'^2 - \mu^2 v^2\right) dt \tag{20}$$

The functional is convex if

$$\int_0^\pi \left(v'^2 - \mu^2 v^2\right) dt \geq 0 \tag{21}$$

for all feasible v-s. But this integral may also be rewritten by Parseval's identity for the Fourier coefficients $\{c\}_n$ of v,

$$\int_0^\pi \left(v'^2 - \mu^2 v^2\right) dt = \frac{\pi}{2} \sum_{n=1}^{\infty} \left(n^2 - \mu^2\right) c_n^2. \tag{22}$$

Thus, L is convex for $\mu \leq 1$ and not convex otherwise. In particular, we observe that L is not convex at the stationary points of L for $\mu = 2, 3, \cdots$.

PS: Show that the stationary points of L for $\mu = 2, 3, \cdots$ are not minima at all.