

QUADRATIC PROGRAMMING BASICS

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Quadratic Programming (QP):

- Common form for a lot of problems
- The iterative step in *Sequential Quadratic Programming (SQP)* methods

THE QP PROBLEM

We are considering problems where the objective function is *quadratic*,

$$q(x) = \frac{1}{2}x'Gx + d'x, \quad G \text{ symmetric.}$$

For the non-constrained problem we know

$$\nabla q(x)' = Gx + d,$$

where

$$x^* = -G^{-1}d \text{ when } G > 0.$$

In general, G will not necessarily be positive definite, not even semi-definite.

The *feasibility domain* Ω is defined in terms of

- **linear equality constraints,**

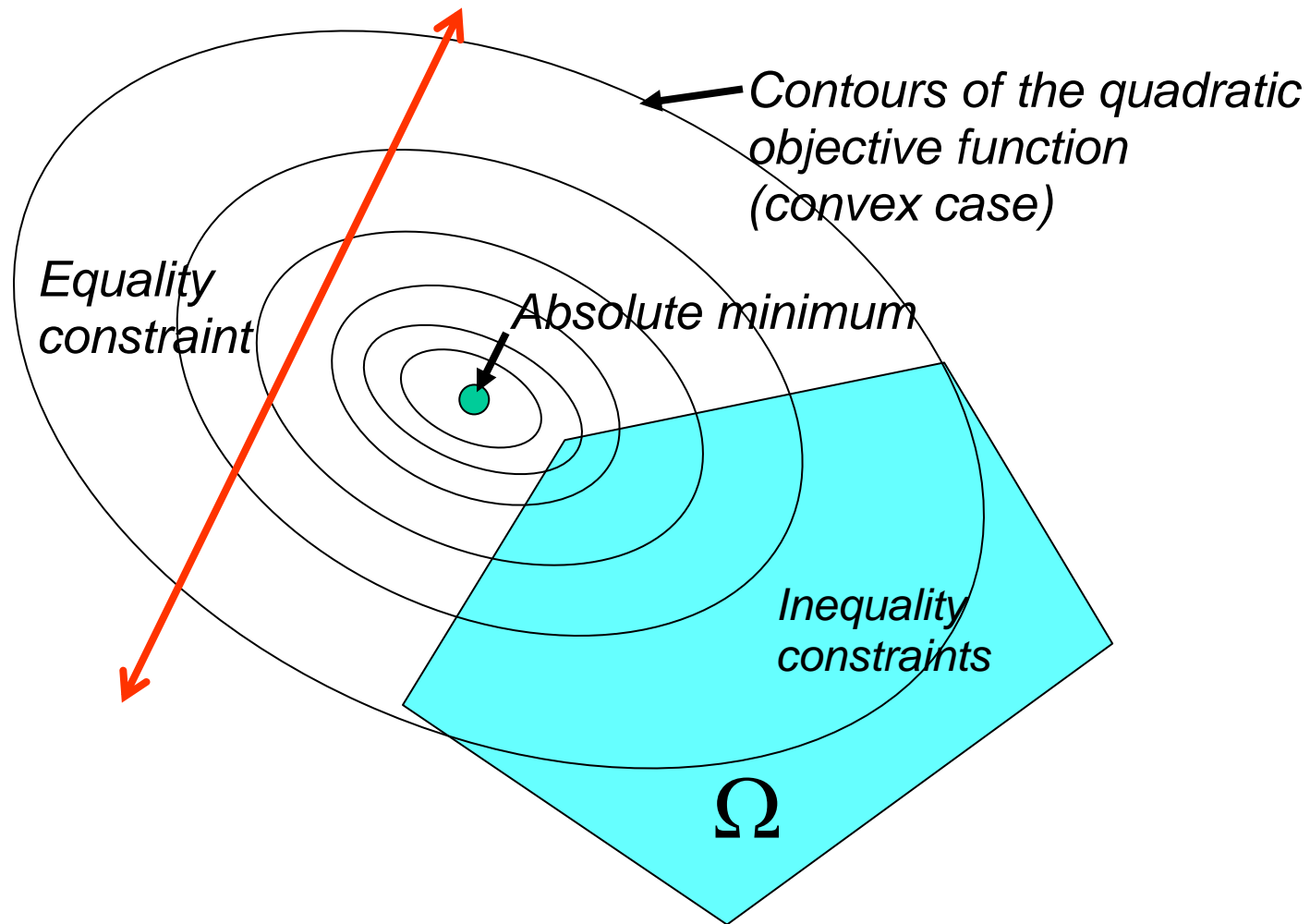
$$a'_i x = b_i, \quad i \in \mathcal{E},$$

- **linear inequality constraints,**

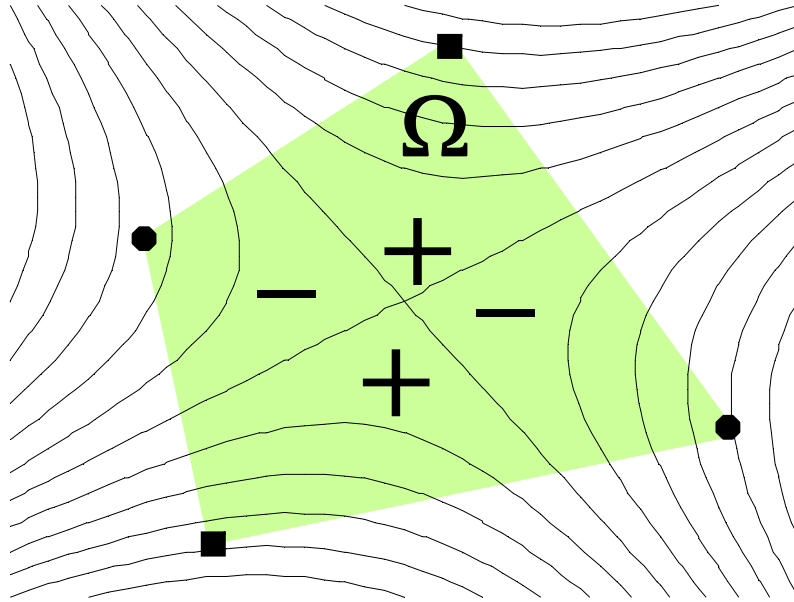
$$a'_i x \geq b_i, \quad i \in \mathcal{I}.$$

NOTE:

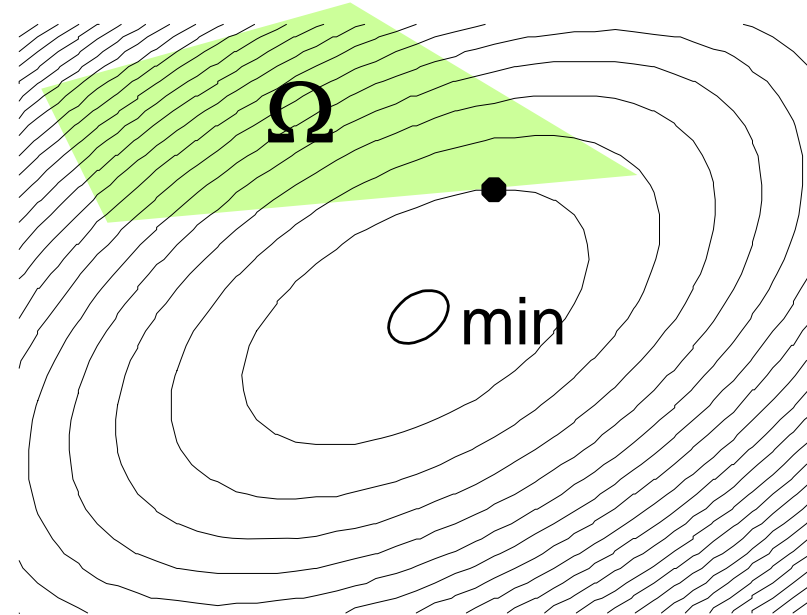
- Ω is convex
- The objective function will be convex if $G \geq 0$.



Indefinite



Positive definite



An indefinite matrix G may lead to several local minima/maxima!

THE QP PROBLEM WITH EQUALITY CONSTRAINTS ONLY

This case can *always* be reduced to the following:

$$\begin{aligned} \min q(x) \\ Ax = b, \\ A \text{ has full rank } r < n. \end{aligned}$$

$$\Omega = \{x; Ax = b\} = \{x_0 + Zu; Ax_0 = b, Z \text{ contains a basis for } \mathcal{N}(A), u \in R^{n-r}\}.$$

1. Solution by Eliminating Unknowns

From the linear system of constraints, express r variables in terms of the remaining $n - r$ unknowns. Insert this into the objective function and solve the unconstrained problem in the remaining $n - r$ variables!

2. Solution by the Null-Space Method

Find an $x_0 \in \Omega$, and a basis for the null space of A . Insert $x = x_0 + Zu$:

$$\begin{aligned} f(u) &= q(x_0 + Zu) = \frac{1}{2}(x_0 + Zu)' G(x_0 + Zu) + d'(x_0 + Zu) \\ &= \frac{1}{2}u' \tilde{G}u + \tilde{d}'u + \text{const.}, \quad \tilde{G} = Z'GZ, \quad \tilde{d} = Z'(Gx_0 + d). \\ \nabla f(u) &= \tilde{G}u + \tilde{d} = 0. \end{aligned}$$

Three cases, depending on \tilde{G} :

- 1) A unique solution if the matrix is positive definite
- 2) Infinitely many solutions if \tilde{G} is singular, as long as it is positive semi-definite and $\tilde{d} \in \mathcal{R}(\tilde{G})$
- 3) No solutions if it is not positive semi-definite (or $\tilde{d} \notin \mathcal{R}(\tilde{G})$)

3. Solving the KKT-equations

The Lagrange function is

$$L(x, \lambda) = \frac{1}{2} x' G x + d' x - \lambda' (b - A x),$$

and hence,

$$\begin{aligned} \nabla_x L(x, \lambda)' &= Gx + d + A' \lambda = 0, \\ Ax &= b. \end{aligned}$$

Collected into a system:

$$\begin{bmatrix} G & A' \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} -d \\ b \end{bmatrix}.$$

Lemma 16.1: *The coefficient matrix of the system is non-singular if A has full rank and G is positive definite on the null-space of A .*

Assume that A has full rank and G is positive definite on the null space of A , that is,

$$(Zu)'G(Zu) = u'Z'GZu = u'\tilde{G}u > 0 \quad \forall u \neq 0.$$

Then (x^*, λ^*) is a unique KKT point and a global minimum (Follows from the Null-Space Method, which then solves a strictly convex problem).

There are many ways of solving the KKT system in this case. However, if x^* is known,

$$\lambda^* = -(AA')^{-1} A(Gx^* + d)$$

If λ^* is known, we reduce the over-determined system

$$Gx = -(d + A'\lambda^*)$$

$$Ax = b$$

to a (non-singular) system with n unknowns (Simple if $G > 0$!)

INEQUALITY CONSTRAINTS

Inequality constraints spoil the elegant theory above completely!

Let us consider the general problem

$$\begin{aligned} \min_x \left\{ \frac{1}{2} x' G x + d' x \right\}, \\ a_i' x - b_i = 0, \quad i \in \mathcal{E}, \\ a_i' x - b_i \geq 0, \quad i \in \mathcal{I}. \end{aligned}$$

We assume that all *equality constraints* are linearly independent.

Let as \mathcal{A} or $\mathcal{A}(x)$ denote the active set of constraints in x

The KKT-conditions:

$$\begin{aligned}Gx + d - \sum_{i \in \mathbf{A}(x)} \lambda_i a_i &= 0, \\a_i' x &= b_i, \quad i \in \mathcal{A}(x), \\a_i' x &> b_i, \quad i \in I \setminus \mathcal{A}(x), \\\lambda_i &\geq 0, \quad i \in I \cap \mathcal{A}(x) \\\lambda_i (a_i' x - b_i) &= 0, \quad i \in \mathcal{E} \cup I\end{aligned}$$

(Recall that the LICQ condition is *not necessary* for *linear* constraints).

If x^* is a KKT-point for the full problem, then x^* and a corresponding subset of the Lagrange multipliers is also a KKT-point for the **reduced problem**

$$\begin{aligned}\min_x \left\{ \frac{1}{2} x' G x + d' x \right\}, \\a_i' x = b_i, \quad i \in \mathcal{A}(x^*)\end{aligned}$$

(Setting $\lambda_i = 0$ for all $i \notin \mathcal{A}(x)$).

- **The reduced problem is a QP problem with equality constraints.**
- If we have an active set \mathcal{A} and have found a KKT-point x^* for the reduced problem, it is easy to check the KKT-conditions for the full problem.
- The next step would be to check the 2nd order conditions: Form the matrix A consisting of the gradients of the active constraints and investigate $Z'GZ$, where Z is a basis of $\mathcal{N}(A)$ (Unless $G \geq 0$, in which case KKT-points are global minima)

ACTIVE SET METHODS

1. Assume we stay at a point $x_0 \in \Omega$.

2. We choose a *working set* \mathcal{W} so that

$$\mathcal{E} \subset \mathcal{W} \subset \mathcal{A}(x_0).$$

3. Let $A_{\mathcal{W}}$ be the corresponding matrix of gradients and solve the (equality constrained) **reduced problem**

$$\begin{aligned} \min q(x_0 + p), \\ A_{\mathcal{W}}(x_0 + p) = b_{\mathcal{W}}. \end{aligned}$$

4. If p turns out to be 0, we have to check whether x_0 could be the full solution.

5. If $p \neq 0$, we consider p as a search direction and determine $\alpha \leq 1$ as the maximum value where

$$x_1 = x_0 + \alpha p \in \Omega.$$

6a. If $\alpha = 1$, we are at a KKT-point x_1 for the reduced problem ($x_1 \in \Omega!$). We then check whether the full KKT-equations are satisfied (in which case we have a solution).

6b. Otherwise, new inequality constraints have become active, which we now **include in** \mathcal{W} , and continue as above from x_1 .

When this stops, we have a KKT-point x^* and an active set \mathcal{W}^* for the reduced problem satisfying

$$Gx^* + d - \sum_{i \in \mathcal{W}^*} \lambda_i^* a_i = 0.$$

7a. Set the Lagrange multipliers for the constraints that are not in \mathcal{W}^* to be 0.

If

$$\lambda_i^* \geq 0$$

for all $i \in \mathcal{W}^* \cap \mathcal{I}$, we have reached a KKT-point for the full problem and that needs to be checked (finished if we have a convex problem).

7b. However, if some of the Lagrangian multipliers are negative, we throw the corresponding constraints out from \mathcal{W}^* and solve the new reduced problem as in (3).

(It may be shown, Theorems 16.5 and 16.6 in N&W, that this will decrease the objective further!).

Note:

- In order to start the method, that is to identify a feasible point $x_0 \in \Omega$, it may be necessary to carry out a Phase 1 problem as in the LP case.
- The Active Set algorithm is listed on p. 472. Numerical aspects shown on pp. 477 – 480.

Algorithm 16.1 (Active-Set Method for Convex QP).

Compute a feasible starting point x_0 ;

Set \mathcal{W}_0 to be a subset of the active constraints at x_0 ;

for $k = 0, 1, 2, \dots$

Solve (16.27) to find p_k ;

if $p_k = 0$

Compute Lagrange multipliers $\hat{\lambda}_i$ that satisfy (16.30),

set $\hat{\mathcal{W}} = \mathcal{W}_k$;

if $\hat{\lambda}_i \geq 0$ for all $i \in \mathcal{W}_k \cap \mathcal{I}$;

STOP with solution $x^* = x_k$;

else

Set $j = \arg \min_{j \in \mathcal{W}_k \cap \mathcal{I}} \hat{\lambda}_j$;

$x_{k+1} = x_k$; $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k \setminus \{j\}$;

else (* $p_k \neq 0$ *)

Compute α_k from (16.29);

$x_{k+1} \leftarrow x_k + \alpha_k p_k$;

if there are blocking constraints

Obtain \mathcal{W}_{k+1} by adding one of the blocking
constraints to \mathcal{W}_{k+1} ;

else

$\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k$;

end (for)

(Copied from Ed. 1. See p. 462, 2nd Ed.)

Example 16.3:

$$\min_x \left\{ (x_1 - 1)^2 + \left(x_2 - \frac{5}{2} \right)^2 \right\}$$

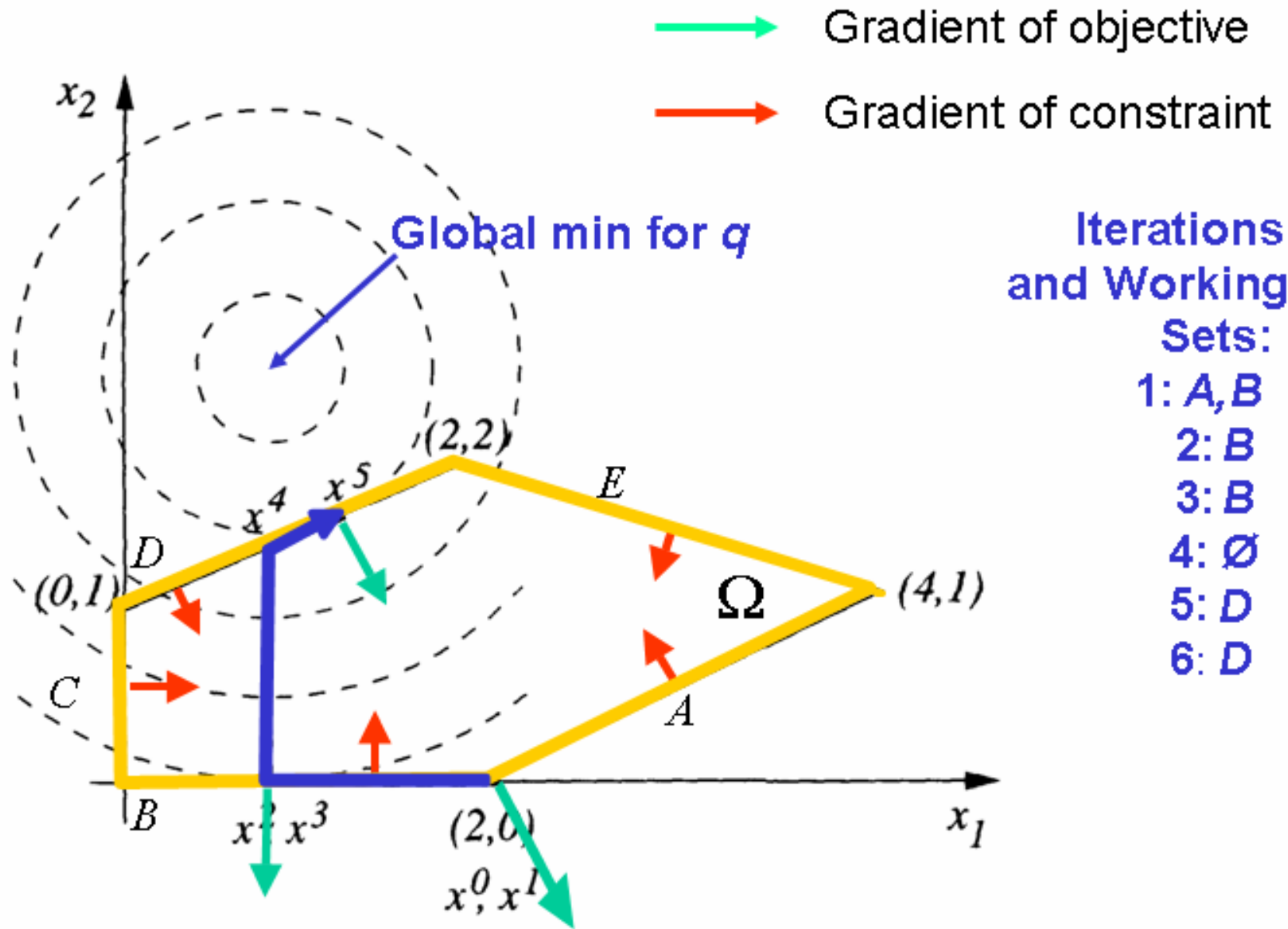
$$x_1 - 2x_2 + 2 \geq 0,$$

$$-x_1 - 2x_2 + 6 \geq 0,$$

$$-x_1 + 2x_2 + 2 \geq 0,$$

$$x_1 \geq 0,$$

$$x_2 \geq 0.$$



THE GRADIENT PROJECTION METHODS

The traditional gradient projection method admits non-linear objective functions as long as the constraints are linear:

$$\begin{aligned} \min f(x), \\ a'_i x = b_i, \quad i \in \mathcal{E}, \\ a'_i x \geq b_i, \quad i \in \mathcal{I}. \end{aligned}$$

We are in a point $x_k \in \Omega$ with a corresponding set of active constraints \mathcal{A}_k and the (full rank) matrix of gradients A_k .

A (local) feasible domain is then

$$\Omega_k = \{x; A_k x = b_k\} = \{x_k + \mathcal{N}(A_k)\}.$$

The gradient in x_k is

$$g_k = \nabla f(x_k)',$$

but in general $x_k - \alpha g_k$ will not be in Ω_k for any $\alpha \neq 0$.

We therefore project the gradient onto $\mathcal{N}(A_k)$ and consider the 1-D problem

$$\min_{\alpha} f(x_k - \alpha P_{\mathcal{N}(A_k)} g_k),$$

$$x_k - \alpha P_{\mathcal{N}(A_k)} g_k \in \Omega.$$

$$P_{\mathcal{N}(A_k)} = I - A_k' (A_k A_k')^{-1} A_k$$

We find the operator by solving the equality constrained QP-problem

$$\min_x \|g - x\|^2,$$

$$Ax = 0.$$

THE NON-LINEAR PROJECTION METHOD

$$\begin{aligned} \min q(x), \\ l \leq x \leq u. \end{aligned}$$

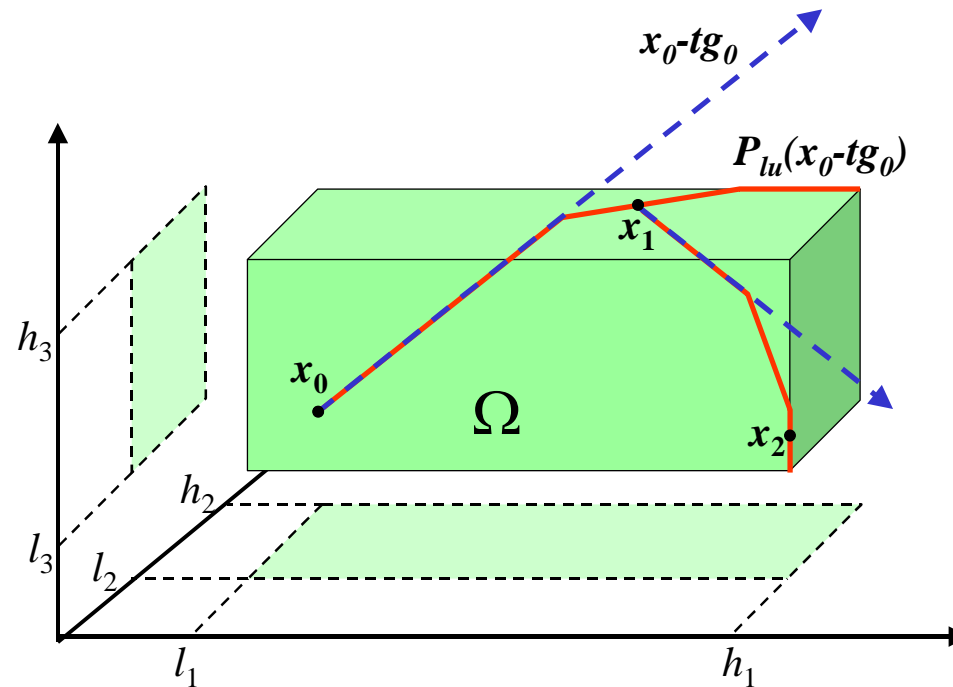
Consider the following (and obvious!) non-linear projection operator

$$P_{lu}(x)_i = \begin{cases} l_i, & x_i \leq l_i \\ x_i, & l_i < x_i < u_i \\ u_i, & u_i \leq x_i \end{cases}$$

We start at x_0 and compute the continuous broken line path

$$x(t) = P_{lu}(x_0 - t\nabla q(x_0)).$$

$$x(t) \in \Omega$$



- Let x_c be the first local minimum along the path.
- From this point the simplest would be to just compute a new gradient and repeat the operation.
- It is also possible to do an approximate Active Set iteration using the already active bounds as the active set, as discussed in N&W p. 480.