Variational Calculus and Sport

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A runner moves because she/he is able to produce a thrust against the ground. However, due to internal friction in the body, the force is a nonlinear function of power (energy per time unit), $W(t)$, produced in the body. We shall assume that $u \propto W^{1/2}$, where u is force measured per unit body mass.

The runner wants to run up a hill with slope α and length L in a certain time T with minimum energy consumption. The speed $y(t)$ is then given by Newton's law,

$$\dot{y}(t) = u(t) - g_\alpha, \tag{1}$$

where $g_\alpha = g \sin\alpha$ (also expressed in gravity force per mass unit). Minimizing the energy consumption, $\int_0^T W(t)dt$, leads to the functional

$$J(y) = \int_0^T u^2(t)dt = \int_0^T (\dot{y}(t) + g_\alpha)^2 \, dt, \tag{2}$$

The runner starts with velocity $y(0) = 0$, and the distance covered should be L,

$$\int_0^T y(t)dt = L. \tag{3}$$

Show that the unique solution of the problem is given by

$$y(t) = \left(3\frac{L}{T^2} + \frac{g_\alpha}{2}\right) t - \frac{3}{4T}\left(\frac{2L}{T^2} + g_\alpha\right) t^2. \tag{4}$$

Another runner wants to traverse an open field from (x_1, y_1) to (x_2, y_2), $x_1 < x_2$, in the shortest possible time. We assume that all paths may be expressed as simple curves in the (x, y)-plane, say $\{x, y(x)\}$, $x_1 < x < x_2$. It's strong wind, and the runner's speed v is dependent on direction relative to the wind, $v = v(y')$. Show that the total time may be expressed as

$$\min_y \int_{x_1}^{x_2} F(y'(x))dx, \tag{5}$$

where

$$F(y') = \frac{\sqrt{1 + y'^2}}{v(y')},$$
$$y(x_1) = y_1, \; y(x_2) = y_2. \tag{6}$$

Determine the optimal path when $F''(z) \neq 0$ for all z.

If we consider the same problem for sailors and windsurfers, there a limit on how much into the wind it is possible to move, say not more than α degrees. Assume that the wind is blowing in the negative x-direction and sketch F as a function of $|y'|$ for this case. Describe the optimal solutions when we neglect the time it takes to turn. Assume that $F''(z) > 0$ at the points where it is defined.

The similar situation also applies for running in a constant sloping terrain, where the speed also depends on the direction one is running.

The terrain in Finland is often rather flat, and Finnish orienteers are known to run long distances at a constant compass heading. Does the Finnish ignorance to small ups and downs really pay?

In this case, the velocity of the runner is influenced by the terrain, described by the surface $z = f(\mathbf{x})$. Introducing the vector path element, $d\mathbf{x}$, we may express the elapsed time, going from a point $\mathbf{x}_1 = (x_1, y_1)$ to another point $\mathbf{x}_2 = (x_2, y_2)$, as

$$\int_{\mathbf{x}_1}^{\mathbf{x}_2} \{a_0 |d\mathbf{x}| + a_1 \nabla f(\mathbf{x}) \cdot d\mathbf{x}\}, \ a_0, a_1 > 0. \tag{7}$$

Here, $a_0 = 1/v_0$, is the inverse of the constant velocity in flat terrain, and the last term is positive (meaning that the velocity is getting smaller) when running uphill, and negative when running downhill. Demonstrate that the Finnish strategy is actually optimal for this case (The argument only needs knowledge about exact differentials).

Finally, a classic problem also often encountered in orienteering. Consider the (x, y)-plane. For $x > 0$ it is an open field, where it is possible to run with large speed v_+. For $x < 0$ there is dense forest with a low running speed v_-. What is the optimal path joining $(-1, 1)$ and $(1, -1)$, and which principle in physics is this analogous to?