Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4180 Optimization Theory
Spring 2013

Exercise set 1

Tutorial: Thursday 24.01 16:15-17:00 in Kjl 4.

1 Consider the problem

$$\min_{(x,y)\in\Omega\subset\mathbb{R}^2}\{x^2+4y^2-2x-8y\}$$

where Ω is defined by the conditions (*constraints*)

$$x,y \geq 0,$$
$$5x+2y \leq 4.$$

(Here and later: $x \geq 0$ for a *vector* in $\mathbb{R}^n$ means $x_i \geq 0$ for $i = 1,2,\ldots,n$. For matrices, $A \geq 0$ means *positive semidefinite*.)

Prove that Ω is convex and f is strictly convex. Do we have a solution and is it unique? Solve the problem in some way or another.

2 **a)** Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function where $\nabla^2 f(x)$ exists and is positive semi-definite, i.e. $\nabla^2 f(x) \geq 0$ at all points. Show that f is convex (*Hint*: Apply the theorem in the Basic Mathematical Tools note about the graph of functions being above their tangent planes).
What can you say if $\nabla^2 f(x)$ is positive definite, that is, $\nabla^2 f(x) > 0$?

b) If $0 \neq a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, the set S defined by

$$S = \{x \mid a^{\mathrm{T}}x \geq \beta\}$$

defines what is called a half-space. Show that it is convex by applying the definition for a convex set. Also show that the set

$$\{x \mid Ax \geq b, x \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m\times n}\}$$

is convex. (*Hint*: Apply the result about the intersection of convex sets.)

c) Determine all minima of the function

$$f(x,y) = x^2+y^2-2yx-2y+2x+5$$

when (x,y) are free to vary over $\mathbb{R}^2$.

d) Sketch how the result of **b)** and **c)** could be used to find all solutions of

$$\min f(x, y),$$

when (x, y) are restricted to satisfy

$$A \begin{bmatrix} x \\ y \end{bmatrix} \geq b, \tag{1}$$

where $f(x, y)$ is the function in **c)**, $A \in \mathbb{R}^{m\times 2}$, $b \in \mathbb{R}^m$, and we know that one of the solutions in **c)** satisfies the inequality (1).
(Show that if we apply what we know from **c)**, all 2-dimensional inequalities in (1) reduce to simple one-dimensional inequalities).

3 A very common optimization problem is the following:

$$\min_{x\in\mathbb{R}^n_+} f(x),$$
$$\mathbb{R}^n_+ = \{x \mid x_i \geq 0, i = 1, \ldots, n\}.$$

Here, $\mathbb{R}^n_+$ is the *non-negative cone* in $\mathbb{R}^n$, and the problem is therefore to find the minimum for x-es where all components are non-negative. We assume that ∇f exists in all points.

Consider the following line-search algorithm: If we are at $x = (x_1, \ldots, x_n)^{\mathrm{T}}$, the search direction, $p = (p_1, \ldots, p_n)^{\mathrm{T}}$, is selected as follows:

$$p_i = \begin{cases} -\frac{\partial f}{\partial x_i}, & \text{if } x_i > 0 \text{ or } \frac{\partial f}{\partial x_i} < 0, \\ 0, & \text{if } x_i = 0 \text{ and } \frac{\partial f}{\partial x_i} \geq 0. \end{cases}$$

a) What are the first-order conditions for a minimum point for this problem? (*Hint*: First-order conditions for a minimum are most easily explained in term of the directional derivatives, which have to be non-negative for all feasible directions d, $\delta f(x, d) = \nabla f(x)^{\mathrm{T}} d \geq 0$. Use this to find the conditions on $\frac{\partial f}{\partial x_i}$, both for interior points and the boundary points.)

b) Show that $p = 0$ at a point satisfying the first-order conditions. (*Hint*: Apply the result from **a)**.)

c) Show that any $p \neq 0$ is a *descent direction.* (The direction d is a descent direction if $\delta f(x, d) < 0$.)