Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4180 Optimization Theory
Spring 2013

Exercise set 8

Tutorial: Thursday 14 16:15-17:00 in Kjl 4.

1 Let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, $\|x\|^2 = \sum_{i=1}^n x_i^2$, $\{y_k\}_{k=1}^K$ be a set of K given vectors in $\mathbb{R}^n$, and define $f(x)$ by

$$f(x) = \frac{1}{2K} \sum_{k=1}^{K} \|x - y_k\|^2.$$

Consider the problem

$$\min f(x)$$

when

$$Ax = b,$$

and A is an $r \times n$ matrix with (full) rank $r < n$.

a) Show, without using part **b)**, that this problem has a unique solution.
Hint: Show that the feasible domain $\Omega = \{x \mid Ax = b\}$ is *convex and non-empty*, and that the function f is *strictly convex* and tends to infinity when $\|x\| \to \infty$.

b) Write down the KKT conditions for the problem and show that the unique solution x^* is

$$x^* = \bar{x} + A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}(b - A\bar{x}),$$
$$\bar{x} = \frac{1}{K} \sum_{k=1}^{K} y_k.$$

c) How will the KKT conditions be modified if some of the equations are replaced by inequalities, e.g.

$$a_i x - b_i \geq 0, \quad i \in \mathcal{I} \subset \{1, \ldots, r\},$$

where a_i is row vector i of A? When will x^* still be a solution?

2 **a)** Write the LP problem

$$\begin{aligned} &\max(-x_1 + x_2), \\ &x_1 \leq 2 + x_2, \\ &6 - x_2 \geq x_1, \end{aligned}$$

on standard form,

$$\begin{aligned} &\min(c^{\mathrm{T}} x), \\ &Ax = b, \\ &x \geq 0. \end{aligned}$$

b) Solve Problem 13.1 (p. 389) in N&W.

3 Determine and solve the *dual* problem to the *primal* problem

$$\begin{aligned} &\min(5x_1 + 3x_2 + 4x_3), \\ &x_1 + x_2 + x_3 = 1, \\ &x_i \geq 0, \quad i = 1, 2, 3. \end{aligned}$$

4 Solve the following LP problem by using the MATLAB Optimization Toolbox:

$$\begin{aligned} &\min(c^{\mathrm{T}} x) \\ &Ax \leq b, \\ &x \geq 0, \end{aligned}$$

where

$$c^{\mathrm{T}} = \begin{bmatrix} -120 & 32 & -48 & -64 \end{bmatrix},$$

$$A = \begin{bmatrix} 3 & 2 & -1 & 4 \\ -1 & 2 & 3 & 12 \\ 4 & 3 & 5 & 21 \\ 8 & 3 & 4 & 5 \\ 6 & 2 & 4 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 42 \\ 36 \\ 45 \\ 28 \\ 14 \end{bmatrix}.$$