Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4180 Optimization Theory
Spring 2013

Exercise set 9

Tutorial: Thursday 21 16:15-17:00 in Kjl 4.

1 (Exam May 2008)

a) What is the content of the duality theorem of linear programming?

b) Show that the following two problems are dual problems (A has full row rank):

$(\mathcal{P})$	$(\mathcal{D})$
$\min_x c^{\mathrm{T}}x$	$\max_\lambda b^{\mathrm{T}}\lambda$
$Ax \geq b, x \geq 0$	$A^{\mathrm{T}}\lambda \leq c, \lambda \geq 0.$

(*Hint*: Consider the KKT equations and start by using (λ, s) as Lagrange multipliers for $(\mathcal{P})$ and (x, u) for $(\mathcal{D})$.)

c) Find the minimum value of

$$2x_1 + 2x_2 + 3x_3 + 2x_4$$

when

$$\begin{aligned} 2x_1 + x_2 + x_3 + 0x_4 &\geq 3 \\ x_1 + 2x_2 + 0x_3 + x_4 &\geq 1 \\ x_i \geq 0, i &= 1, \dots, 4. \end{aligned}$$

Also, find x.

2 Solve the problem

$$\begin{aligned} \min_x (2x_1 + 3x_2 + 4x_1^2 + 2x_1x_2 + x_2^2) \\ x_1 - x_2 \geq 0 \\ x_1 + x_2 \leq 4 \\ x_1 \leq 3 \\ x_2 \geq 0 \end{aligned}$$

graphically. Verify the KKT conditions at the solution.

3 N&W 2nd edition, exercise 16.2, p. 493.

4 (Exam June 2006) Given the following problem:

$$\min_{x \in \mathbb{R}^n} q(x) \tag{1}$$
$$\text{where } a_i^{\mathrm{T}} x \geq b_i, \qquad i = 1, 2, \ldots, m,$$

with $q(x) := \frac{1}{2} x^{\mathrm{T}} G x + x^{\mathrm{T}} d$ and G a symmetric, positive definite matrix.

a) Write down the Karush-Kuhn-Tucker conditions for the problem (but do not solve them). Assume that $x^\star$ is a solution to the KKT conditions. Is $x^\star$ then a global minimum of (1)?

b) Solve the problem

$$\min\, (x_1 - 1)^2 + (x_2 - 1)^2, \tag{2}$$

where

$$\begin{aligned} -x_1 - 2x_2 &\geq -2 \\ -2x_1 \quad &\geq -3 \\ x_1, x_2 \quad &\geq 0. \end{aligned}$$

Hint: Start with a sketch of the admissible domain and the level curves of $q(x)$.

The remainder of this problem is about constructing an iterative method for solving the general quadratic problem (1).

Given an admissible point x_0, let $\mathcal{W}$ be a given set of *active constraints* in x_0, such that $\mathcal{W} \subset \mathcal{A}(x_0)$.

c) Find a solution p of the reduced problem

$$\min_{p} q(x_0 + p)$$
$$\text{where } a_i^{\mathrm{T}} (x_0 + p) = b_i, \qquad i \in \mathcal{W},$$

assuming that the a_i are linearly independent for $i \in \mathcal{W}$.
Hint: Show first that this is equivalent to finding a minimum of $p^{\mathrm{T}} G p/2 + p^{\mathrm{T}} (Gx_0 + d)$ with the constraints $a_i^{\mathrm{T}} p = 0$ for $i \in \mathcal{W}$.

d) Assume that the solution p of c) is not 0. Find an expression for the maximum value α such that $x_0 + \alpha p$ is an admissible point. The next step in the iteration is then given by

$$x_1 := x_0 + \min\{\alpha, 1\} \cdot p.$$

Explain why.

e) Execute one iteration step (points c) and d)) for problem (2). Start with $x_0 := [3/2, 0]^{\mathrm{T}}$. Choose $\mathcal{W}$ yourself.
Note: Even if you have not found formal solutions of the general problem in c) and d) it can well happen that you will find a solution to the specific problem (2).

f) The points c) and d) are part of an *active set method* for quadratic problems. To complete the algorithm, the following questions have to be answered:

- Is x_1 a solution?

- If not, how should we choose $\mathcal{W}$ in the next iteration?

Explain how these questions can be answered.

5 The following problem is copied from T.L. Saaty and J. Bram: *Nonlinear Mathematics*, p. 144. This is a little challenge!

Solve

$$\min f(x), \quad f(x) = x_1^3 - 6x_1^2 + 11x_1 + x_3,$$
$$-x_1^2 - x_2^2 + x_3^2 \geq 0,$$
$$x_1^2 + x_2^2 + x_3^2 - 4 \geq 0,$$
$$-x_3 + 5 \geq 0,$$
$$x_1, x_2, x_3 \geq 0.$$

Observe that f is independent of x_2.

a) Show that $(0, \sqrt{2}, \sqrt{2})$ is a KKT point.

b) Do some numerical experiments using the MATLAB Optimization Toolbox function `fmincon`. Suggested code:

```
x0 = [2.1, 0, 2.1]';
x = fmincon(@SBfunction, x0, [], [], [], [], [], [], @constraints);
```

Function:

```
function f = SBfunction(x)
    f = x(1)^3 - 6*x(1)^2 + 11*x(1) + x(3);
end
```

Constraints:

```
function [c, ceq] = constraints(x)
    % Nonlinear inequalities (note sign-convention!)
    c(1) = x(1)^2 + x(2)^2 - x(3)^2;
    c(2) = -x(1)^2 - x(2)^2 - x(3)^2 + 4;
    c(3) = x(3) - 5;
    c(4) = -x(1);
    c(5) = -x(2);
    c(6) = -x(3);
    ceq = [];
end
```

c) The book states that *"This problem actually has another local solution"*. Is this really true? Use MATLAB for experiments, but try to prove your claims. (This seems to be a challenge!)